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# Superposition of Wave Functions in the G-Qubit Theory

Alexander SOIGUINE[1]

[1] SOiGUINE Quantum Computing, USA

Correspondence: Alexander SOIGUINE, SOiGUINE Quantum Computing, Aliso Viejo, CA 92656, USA. Website: https://soiguine.com; E-mail: alex@soiguine.com

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**Abstract**

Quantum computing rests upon two theoretical pillars: superposition and entanglement. But some physicists say that this is a very shaky foundation and quantum computing success will have to be based on a different theoretical foundation. The g-qubit theory supports that point of view. Current article is the first one of two and about the superposition principle. Quantum superposition principle states that any two quantum wave functions can be superposed, and the result be another valid wave function. It specifically refers to linearity of the Schrodinger equation. In the g-qubit theory quantum wave functions are identified by points on the surface of three-dimensional sphere $\mathbb{S}^3$. That gives different, more physically feasible, not mysterious, explanation of what the superposition is.

**Keywords:** geometric algebra, wave functions, observables, measurements

## 1. Introduction. Arithmetic of the $G_3^+$ elements

For two arbitrary elements of $G_3^+$, $g_1 = \alpha_1 + I_{S_1}\beta_1$ and $g_2 = \alpha_2 + I_{S_2}\beta_2$, see [1], with generally different bivector planes of $I_{S_1}$ and $I_{S_2}$, both $I_{S_1}$ and $I_{S_2}$ assumed to be unit value ones, we have:

$$g_1 + g_2 = (\alpha_1 + \alpha_2) + \left(I_{S_1}\beta_1 + I_{S_2}\beta_2\right)$$

The sum of two bivectors $I_{S_1}\beta_1 + I_{S_2}\beta_2$ is a bivector lying in plane different from both $I_{S_1}$ and $I_{S_2}$ (see Figure 1.1)

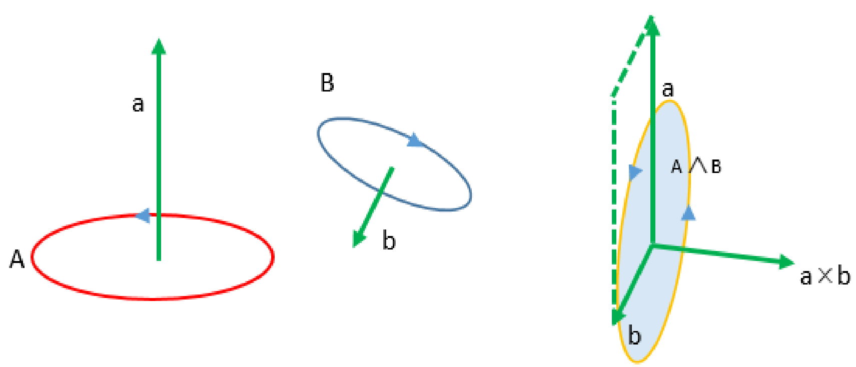


Figure 1.1. Addition of two bivectors using corresponding normal vectors in the right-screw oriented space

To make the result more convenient for calculations expand bivectors $I_{S_1}$ and $I_{S_2}$ in a basis $\{B_1, B_2, B_3\}$ - an arbitrary triple of unit value mutually orthogonal bivectors in three dimensions satisfying, with not critical assumption of right-hand screw orientation $B_1B_2B_3 = 1$, the multiplication rules (see Figure 1.2):

$$B_1B_2 = -B_3,\ B_1B_3 = B_2,\ B_2B_3 = -B_1 \text{(Note 1)}$$

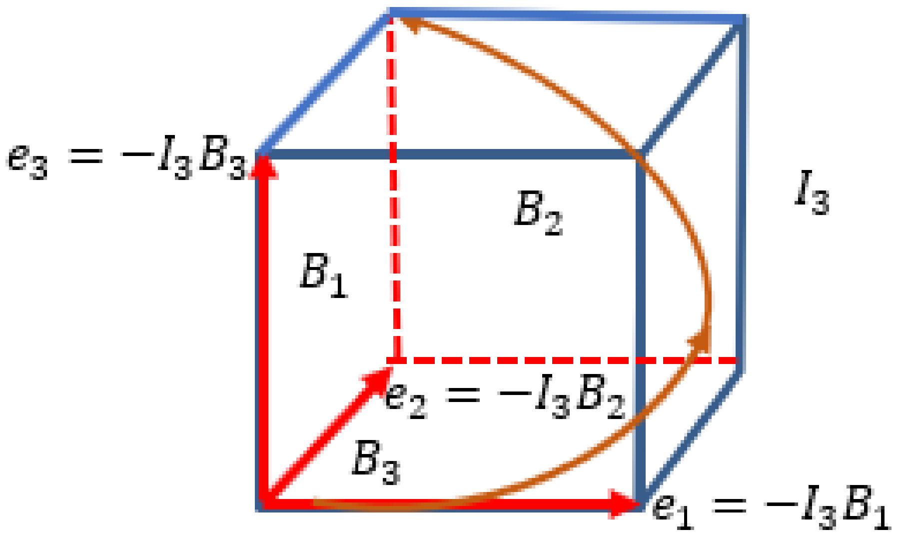

Figure 1.2. Basis of bivectors, vectors associated with the bivectors, and unit value pseudoscalar

Thus:

$$I_{S_1} = b_1^1 B_1 + b_1^2 B_2 + b_1^3 B_3$$
$$I_{S_2} = b_2^1 B_1 + b_2^2 B_2 + b_2^3 B_3$$

Then:

$$\begin{aligned} I_{S_1}\beta_1 + I_{S_2}\beta_2 &= \beta_1(b_1^1 B_1 + b_1^2 B_2 + b_1^3 B_3) + \beta_2(b_2^1 B_1 + b_2^2 B_2 + b_2^3 B_3) \\ &= (\beta_1 b_1^1 + \beta_2 b_2^1)B_1 + (\beta_1 b_1^2 + \beta_2 b_2^2)B_2 + (\beta_1 b_1^3 + \beta_2 b_2^3)B_3 \end{aligned}$$

and finally

$$g_1 + g_2 = (\alpha_1 + \alpha_2) + (\beta_1 b_1^1 + \beta_2 b_2^1)B_1 + (\beta_1 b_1^2 + \beta_2 b_2^2)B_2 + (\beta_1 b_1^3 + \beta_2 b_2^3)B_3$$

The length of the sum, due to the unit values of $I_{S_1}$ and $I_{S_2}$, that's $(b_1^1)^2 + (b_1^2)^2 + (b_1^3)^2 = 1$ and $(b_2^1)^2 + (b_2^2)^2 + (b_2^3)^2 = 1$, can be written as

$$\sqrt{(\alpha_1 + \alpha_2)^2 + \beta_1^2 + \beta_2^2 + 2\beta_1\beta_2(s_1 \cdot s_2)}$$

where $s_1 = b_1^1 e_1 + b_1^2 e_2 + b_1^3 e_3$ and $s_2 = b_2^1 e_1 + b_2^2 e_2 + b_2^3 e_3$ (Note 2) are vectors dual to $I_{S_1}$ and $I_{S_2}$, $s_1 = -I_3 I_{S_1}$, $s_2 = -I_3 I_{S_2}$.

Take an arbitrary $g = \alpha + I_S\beta$. Rewrite it in the form $g = \sqrt{\alpha^2 + \beta^2}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} + \frac{\beta}{\sqrt{\alpha^2+\beta^2}} I_S\right)$. Since $\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)^2 + \left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)^2 = 1$ we can define $\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \equiv \cos\varphi$ and $\frac{\beta}{\sqrt{\alpha^2+\beta^2}} \equiv \sin\varphi$. In that way we can write $g = \sqrt{\alpha^2 + \beta^2}(\cos\varphi + \sin\varphi\, I_S)$.

Expand $I_S = b_1 B_1 + b_2 B_2 + b_3 B_3$. Directional cosines $\{b_1, b_2, b_3\}$ define orientation of unit radius disc $I_S$ in three-dimensional space.

Consider sphere of radius $\sqrt{\alpha^2 + \beta^2}$ and intersect it by a plane parallel to $I_S$. Choose some two-dimensional coordinate system $\{x, y\}$ in that plane. In this way $g$ is fully identified by the sphere of radius $\sqrt{\alpha^2 + \beta^2}$, vector $s$ of the length equal to the sphere radius $\sqrt{\alpha^2 + \beta^2}$ with directional cosines $\{b_1, b_2, b_3\}$, and angle of rotation around $s$ in the coordinate system $\{x, y\}$: $\varphi = cos^{-1}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)$ (see Fig.1.3.)

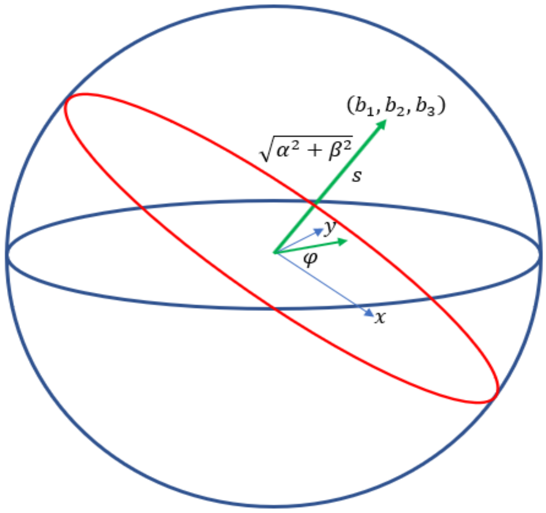


Figure 1.3. Geometric representation of a $G_3^+$ element

The sum of two $G_3^+$ elements with vectors $s_1$ and $s_2$ associated with unit value bivectors $I_{s_1}$ and $I_{s_2}$ has the associated vector $s = s_1 + s_2$, (see Fig.1.4)

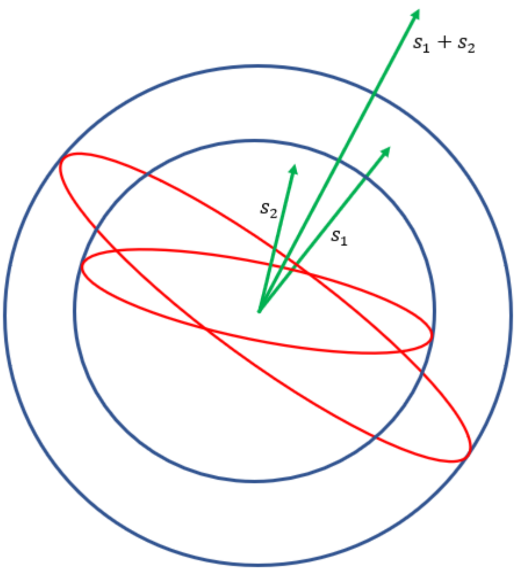


Figure 1.4. Geometric representation of addition

Torsion angle in plane orthogonal to $s_1 + s_2$ is $cos^{-1}\left(\frac{\alpha_1+\alpha_2}{\sqrt{(\alpha_1+\alpha_2)^2+\beta_1^2+\beta_2^2+2\beta_1\beta_2(s_1\cdot s_2)}}\right)$.

## 2. Superposition of Two Basic Wave Functions Corresponding to $|0\rangle$ and $|1\rangle$

The quantum mechanical qubit state, $|\psi\rangle = z_1|0\rangle + z_2|1\rangle$, is linear combination of two basis states $|0\rangle$ and $|1\rangle$. In more details:

$$|\psi\rangle = \begin{pmatrix} \alpha + i\beta_1 \\ \beta_3 + i\beta_2 \end{pmatrix}$$

There exist infinite number of options to select triple $\{B_1, B_2, B_3\}$. Thus, the procedure of recovering a g-qubit associated with $|\psi\rangle = z_1|0\rangle + z_2|1\rangle$ is the following one:
It is necessary, see [2], [3], firstly, to define bivector $B_{i_1}$ in three dimensions identifying the torsion plane. Secondly, choose another bivector $B_{i_2}$ orthogonal to $B_{i_1}$. The third bivector $B_{i_3}$ , orthogonal to both $B_{i_1}$ and $B_{i_1}$, is then defined by the first two by orientation (handedness, right screw in the used case ): $I_3 B_{i_1} I_3 B_{i_2} I_3 B_{i_3} = I_3$.
Wave functions in the suggested theory are operators acting through measurements on observables:

$$\overline{(\alpha + I_S\beta)} C(\alpha + I_S\beta)$$

For any wave function $\alpha + B_{i_1}\beta_i,\ i = 1,2,3$, corresponding to $|0\rangle$ (assuming $\alpha^2 + {\beta_i}^2 = 1$) we get:

$$(\alpha - B_{i_1}\beta_i) B_{i_1} (\alpha + B_{i_1}\beta_i) = (\alpha^2 + {\beta_i}^2) B_{i_1} = B_{i_1}$$

For the wave functions $\beta_{(i+2)mod3} B_{(i+2)mod3} + \beta_{(i+1)mod3} B_{(i+1)mod3}$, , $i = 1,2,3$, corresponding to $|1\rangle$ (with the agreement $3mod3 = 3$) the value of observable $B_{i_1}$is (with same assumption ${\beta_{(i+2)mod3}}^2 + {\beta_{(i+1)mod3}}^2 = 1$):

$$\left(-\beta_{(i+2)mod3} B_{(i+2)mod3} - \beta_{(i+1)mod3} B_{(i+1)mod3}\right) B_{i_1} \left(\beta_{(i+2)mod3} B_{(i+2)mod3} + \beta_{(i+1)mod3} B_{(i+1)mod3}\right)$$
$$= -\left({\beta_{(i+2)mod3}}^2 + {\beta_{(i+1)mod3}}^2\right) B_{i_1} = -B_{i_1}$$

Let us take an arbitrary bivector observable expanded in basis $\{B_1, B_2, B_3\} \equiv \{B_{i_1}, B_{i_2}, B_{i_3}\}$:

$$C = C_1 B_1 + C_2 B_2 + C_3 B_3$$

The result of measurement by wave function corresponding to $|0\rangle$ is:

$$(\alpha - \beta_1 B_1) C (\alpha + \beta_1 B_1) = C_1 B_1 + [C_2(\alpha^2 - \beta_1^2) - 2C_3\alpha\beta_1] B_2 + [C_3(\alpha^2 - \beta_1^2) + 2C_2\alpha\beta_1] B_3 = C_1 B_1 + (C_2 \cos 2\varphi - C_3 \sin 2\varphi) B_2 + (C_2 \sin 2\varphi + C_3 \cos 2\varphi) B_3 \quad , \tag{2.1}$$

using parametrization $\alpha = \cos\varphi,\ \beta_1 = \sin\varphi$.
The result of measurement by wave function corresponding to $|1\rangle$ is:

$$(-\beta_2 B_2 - \beta_3 B_3) C (\beta_2 B_2 + \beta_3 B_3) = -C_1 B_1 + [C_2(\beta_2^2 - \beta_3^2) + 2C_3\beta_2\beta_3] B_2 + [2C_2\beta_2\beta_3 - C_3(\beta_2^2 - \beta_3^2)] B_3 = -C_1 B_1 + (C_2 \cos 2\theta + C_3 \sin 2\theta) B_2 + (C_2 \sin 2\theta - C_3 \cos 2\theta) B_3, \tag{2.2}$$

with $\beta_2 = \cos\theta,\ \beta_3 = \sin\theta$.
This is a deeper result compared with conventional quantum mechanics where

$$|0\rangle = \binom{\alpha + i\beta_1}{0} \text{ and } |1\rangle = \binom{0}{\beta_3 + i\beta_2}$$

Conclusion:

- Measurement of observable $C = C_1 B_1 + C_2 B_2 + C_3 B_3$ by any wave function corresponding to $|0\rangle$ is bivector with the $B_1$ component equal to unchanged value $C_1$. The $B_2$ and $B_3$ components of the result of measurement are equal to $B_2$ and $B_3$ components of $C$ rotated by angle $2\varphi$ defined by $\alpha = \cos\varphi,\ \beta_1 = \sin\varphi$ where plane of rotation is $B_1$.
- Measurement of observable $C = C_1 B_1 + C_2 B_2 + C_3 B_3$ by any wave function corresponding to $|1\rangle$ is bivector with the $B_1$ component equal to flipped value $-C_1$. The $B_2$ and $B_3$ components of the result of measurement are equal to $B_2$ and $B_3$ components of $C$ rotated by angle $2\theta$ defined by $\beta_2 = \cos\theta,\ \beta_3 = \sin\theta$ where plane of rotation is $B_1$ but direction of rotation is opposite to the case of $|0\rangle$.

If we denote by $so_{|0\rangle}$ and $so_{|1\rangle}$ arbitrary wave functions represented in $G_3^+$ by $\alpha + \beta_1 B_1$ and $\beta_2 B_2 + \beta_3 B_3 = (\beta_3 + \beta_2 B_1) B_3$ they only differ by factor $B_3$ in $so_{|1\rangle}$, thus for the measurement by them we have:

$$\overline{so_{|1\rangle}} C so_{|1\rangle} = \overline{B_3}\,\overline{so_{|0\rangle}} C so_{|0\rangle} B_3$$

That simply means that the measurement on the left side is received from $\overline{so_{|0\rangle}} C so_{|0\rangle}$ by its flipping in plane $B_3$ (see Figure 2.1)

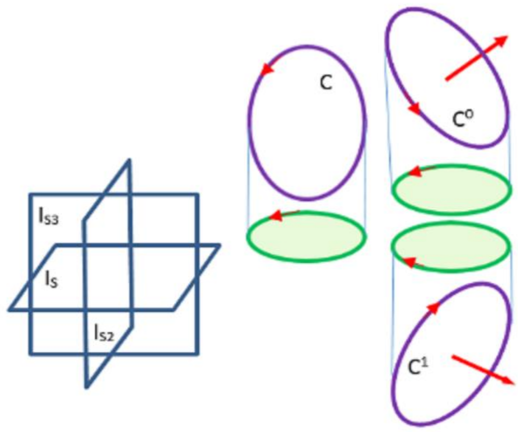


Figure 2.1. Difference in measuring by $so_{|0\rangle}$ and $so_{|1\rangle}$

Probabilities of the results of measurements are measures of wave functions on $\mathbb{S}^3$ surface giving considered results.

Suppose we are interested in the probability of the result of measurement in which the observable component $C_1B_1$ does not change. This is relative measure of wave functions $\sqrt{\alpha^2+\beta_1^2}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta_1^2}}+\frac{\beta_1}{\sqrt{\alpha^2+\beta_1^2}}B_1\right)$ in the measurements:

$$\sqrt{\alpha^2+\beta_1^2}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta_1^2}}-\frac{\beta_1}{\sqrt{\alpha^2+\beta_1^2}}B_1\right)C\sqrt{\alpha^2+\beta_1^2}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta_1^2}}+\frac{\beta_1}{\sqrt{\alpha^2+\beta_1^2}}B_1\right) \quad (2.3)$$

That measure is equal to $\alpha^2+\beta_1^2$, that is equal to $z_1^2$ in the down mapping from $G_3^+$ to Hilbert space of $z_1|0\rangle + z_2\ |1\rangle$. Thus, we have clear explanation of common quantum mechanics wisdom on "probability of finding system in state $|0\rangle$".

Similar calculations explain correspondence of $\beta_3^2+\beta_2^2$ to $z_2^2$ in the qubit $z_1|0\rangle + z_2 1\rangle$ when the component $C_1B_1$ in measurement just got flipped.

Let us consider superposition of $\alpha+\beta_1B_1$ and $\beta_2B_2+\beta_3B_3$ with some coefficients $p_1$ and $p_2$(Note 3),

$$p_1(\alpha+\beta_1B_1)+p_2(\beta_2B_2+\beta_3B_3),$$

and measuring by it of $C = C_1B_1+C_2B_2+C_3B_3$.

$$\begin{aligned}
&[p_1(\alpha-\beta_1B_1)+p_2(-\beta_2B_2-\beta_3B_3)]C[p_1(\alpha+\beta_1B_1)+p_2(\beta_2B_2+\beta_3B_3)]\\
&= p_1(\alpha-\beta_1B_1)Cp_1(\alpha+\beta_1B_1)+p_2(-\beta_2B_2-\beta_3B_3)Cp_2(\beta_2B_2+\beta_3B_3)\\
&+ p_2(-\beta_2B_2-\beta_3B_3)Cp_1(\alpha+\beta_1B_1)+p_1(\alpha-\beta_1B_1)Cp_2(\beta_2B_2+\beta_3B_3)\\
&= p_1(\alpha-\beta_1B_1)Cp_1(\alpha+\beta_1B_1)+p_2(-\beta_2B_2-\beta_3B_3)Cp_2(\beta_2B_2+\beta_3B_3)\\
&+ p_1(\alpha-\beta_1B_1)Cp_1(\alpha+\beta_1B_1)p_1(\alpha-\beta_1B_1)p_2(\beta_2B_2+\beta_3B_3)\\
&+ p_2(-\beta_2B_2-\beta_3B_3)Cp_2(\beta_2B_2+\beta_3B_3)p_2(-\beta_2B_2-\beta_3B_3)p_1(\alpha+\beta_1B_1)
\end{aligned}$$

It follows from this formula that the result of measurement by wave function $p_1(\alpha+\beta_1B_1)+p_2(\beta_2B_2+\beta_3B_3)$ makes the $C_1B_1$ component unchanged and two other components rotated around the normal to $B_1$, see (2.1) and (2.3), with probability ${p_1}^2$ (item $p_1(\alpha-\beta_1B_1)Cp_1(\alpha+\beta_1B_1)$). Then it just flips the $C_1B_1$ component and two other components rotated around the normal to $B_1$, but in opposite direction see (2.2) with probability ${p_2}^2$ (item $p_2(-\beta_2B_2-\beta_3B_3)Cp_2(\beta_2B_2+\beta_3B_3)$). Other two items are correspondingly the first above result subjected to

Clifford (parallel) translation on $\mathbb{S}^3$ by $p_1p_2(\alpha - \beta_1 B_1)(\beta_2 B_2 + \beta_3 B_3)$ and the second result subjected to opposite Clifford translation $p_1p_2(-\beta_2 B_2 - \beta_3 B_3)(\alpha + \beta_1 B_1)$. They are neither (2.1) nor (2.2) and their probabilities to make $C_1B_1$ unchanged or flipped are zero. Though, they give two other different available measurement results.

## 3. Superposition of two arbitrary wave functions

Any arbitrary $G_3^+$ wave function $\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$ can be rewritten either as 0-type wave function or 1-type wave function:

$$\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 = \alpha + I_{S(\beta_1,\beta_2,\beta_3)}\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2},$$
$$\text{where } I_{S(\beta_1,\beta_2,\beta_3)} = \frac{\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3}{\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}}, \qquad \text{0-type,}$$

or

$$\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 = (\beta_3 + \beta_2 B_1 - \beta_1 B_2 - \alpha B_3)B_3 = \left(\beta_3 + I_{S(\beta_2,-\beta_1,-\alpha)}\sqrt{\alpha^2 + \beta_1^2 + \beta_2^2}\right)B_3,$$

where $I_{S(\beta_2,-\beta_1,-\alpha)} = \frac{\beta_2 B_1 - \beta_1 B_2 - \alpha B_3}{\sqrt{\alpha^2+\beta_1^2+\beta_2^2}}$, 1-type.

All that means that any $G_3^+$ wave function $\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$ measuring observable $C_1B_1 + C_2B_2 + C_3B_3$ does not change the observable projection onto plane of $I_{S(\beta_1,\beta_2,\beta_3)} = \frac{\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3}{\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}}$ and just flips the observable projection onto plane $I_{S(\beta_2,-\beta_1,-\alpha)} = \frac{\beta_2 B_1 - \beta_1 B_2 - \alpha B_3}{\sqrt{\alpha^2+\beta_1^2+\beta_2^2}}$.

Take two arbitrary wave functions and rewrite the first one as 0-type wave function and the second one as 1-type wave function. Then all the results of Sec.2 become applicable for their superposition. It will follow that there will be a result of measurement

$$p_1^2\left(\alpha - I_{S(\beta_1,\beta_2,\beta_3)}\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}\right)C\left(\alpha + I_{S(\beta_1,\beta_2,\beta_3)}\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}\right)$$

not changing the projection of $C$ onto plane of $I_{S(\beta_1,\beta_2,\beta_3)}$ and keeping probability $p_1^2$; plus, result of measurement

$$p_2^2\left(-B_3\left(\beta_3 - I_{S(\beta_2,-\beta_1,-\alpha)}\sqrt{\alpha^2 + \beta_1^2 + \beta_2^2}\right)\right)C\left(\left(\beta_3 + I_{S(\beta_2,-\beta_1,-\alpha)}\sqrt{\alpha^2 + \beta_1^2 + \beta_2^2}\right)B_3\right)$$

just flipping projection of $C$ in plane of $I_{S(\beta_2,-\beta_1,-\alpha)}$ and keeping probability $p_2^2$. Two other results represent the first two subjected to Clifford (parallel) translations on the sphere $\mathbb{S}^3$ *by*

$$p_1p_2\left(\alpha - I_{S(\beta_1,\beta_2,\beta_3)}\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}\right)\left(\beta_3 + I_{S(\beta_2,-\beta_1,-\alpha)}\sqrt{\alpha^2 + \beta_1^2 + \beta_2^2}\right)$$

and

$$p_1p_2\left(\beta_3 - I_{S(\beta_2,-\beta_1,-\alpha)}\sqrt{\alpha^2 + \beta_1^2 + \beta_2^2}\right)\left(\alpha + I_{S(\beta_1,\beta_2,\beta_3)}\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}\right)$$

correspondingly.

## 4. Conclusions

Superposition of any two wave functions in the frame of g-qubit theory gives another wave function the result of measurement by which is more complicated than in conventional quantum mechanics. In addition to the two results of measurements coming from composed items of the wave functions there appear two additional items which are Clifford (parallel) translations of the first two results in opposite directions on the sphere $\mathbb{S}^3$.

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**Notes**

Note 1. Opposite orientation $B_1 B_2 B_3 = -1$ can be equivalently used

Note 2. Length $\sqrt{\alpha^2 + \beta^2}$ of vector $s$ associated with $g = \alpha + I_S \beta$ will be also called *module* of $g$ and denoted $|g|$

Note 3. Due to orthogonality of $\alpha + \beta_1 B_1$ and $\beta_2 B_2 + \beta_3 B_3$ in Euclidean sense on the three-sphere $s_1 \cdot s_2 = 0$ in this case