Original Paper

Explaining Double Split Experiment with Geometrical Algebra

Formalism

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Abstract
The Geometric Algebra formalism opens the door to developing a theory upgrading conventional quantum mechanics. Generalizations, stemming from implementation of complex numbers as geometrically feasible objects in three dimensions; unambiguous definition of states, observables, measurements bring into reality clear explanations of conventional weird quantum mechanical features, particularly the results of double split experiments where particles create diffraction patterns inherent to wave diffraction. This weirdness of the double split experiment is milestone of all further difficulties in interpretation of quantum mechanics.

Keywords
geometric algebra, quantum states, observables, measurements

1. Introduction. Working with G-qubits Instead of Qubits
Theory of upgrading conventional quantum mechanics has been under development (Soiguine, 1996, 2014, 2015, 2020).

The main novel features are:

- Replacing complex numbers with elements of even subalgebra of geometric algebra in three dimensions, that’s by elements of the form “scalar plus bivector”.
- The objects identifying physical media are of the same structure: explicitly defined plane along with angle of rotation in that plane.
- Operators acting on the objects are operators of rotation having the same structure: scalar plus bivector. That is the measurement operation.
- Mapping of operator to the result of measurement, that is collapse, creates a “particle”.

46
- The operators can be identified by points on the three-sphere $\mathbb{S}^3$ and are connected, due to hedgehog theorem, by Clifford translations. Modifying the operators due to Clifford translation is identified by the generalization of Schrödinger equation containing unit bivectors in three dimensions instead of formal imaginary unit.

Qubits, identifying states in conventional quantum mechanics, mathematically are elements of two-dimensional complex spaces:

$$\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right), \|x_1 + iy_1\|^2 + \|x_2 + iy_2\|^2 = 1,$$

unit value element of $\mathbb{C}^2$.

Imaginary unit $i$ is used formally, without geometrical identification in three dimensions. In another accepted notations a qubit is:

$$\mathbb{C}^2 \ni \left(\begin{array}{c} z_1 \\ z_2 \end{array}\right) = z_1 \left(\begin{array}{c} 1 \\ 0 \end{array}\right) + z_2 \left(\begin{array}{c} 0 \\ 1 \end{array}\right) = z_1 |0\rangle + z_2 |1\rangle$$

In the suggested formalism complex numbers $x + iy$ are replaced with elements of $G^+_3$, subalgebra of $G_3$.

The $G_3$ elements of the form $M_3 = \alpha + I_S \beta$, where $I_S$ is some unit bivector arbitrary placed in three-dimensional space, comprise so called even subalgebra of algebra $G_3$. This subalgebra is denoted by $G^+_3$ (Soiguine, 2015, 2020). Elements of $G^+_3$ can be depict as in Figure 1.

![Figure 1. An Element of $G^+_3$](image)

Unit value elements of $G^+_3$, $\alpha^2 + \beta^2 = 1$, will be called $g$-qubits. The wave functions (states) implemented as $g$-qubits store much more information than qubits, see Figure 2.
2. Lift of Qubits to G-qubits, Fiber Bundles and Probabilities

Take right-hand screw oriented basis \( \{ B_1, B_2, B_3 \} \) of unit value bivectors, with the multiplication rules \( B_1B_2 = -B_3 \), \( B_1B_3 = B_2 \), \( B_2B_3 = -B_1 \), \( I_3B_1I_3B_2I_3B_3 = I_3 \) (or equivalently \( B_1B_2B_3 = 1 \)), where \( I_3 \) is oriented unit value volume in three dimensions named also pseudoscalar.

Quantum mechanical qubit state, \( |\psi\rangle = z_1|0\rangle + z_2|1\rangle \), is linear combination of two basis states \( |0\rangle \) and \( |1\rangle \). In the \( G^+_3 \) terms these two states correspond to the following classes of equivalence in \( G^+_3 \), depending on which basis bivector is selected as complex plane:

If \( B_1 \) is taken as complex plane, then
- State \( |0\rangle \) has fiber (level set) of the \( G^+_3 \) elements \( so(\alpha, \beta, S)_{10} \) (0-type \( G^+_3 \) states):
  - \( \alpha + \beta_1B_1, \alpha^2 + \beta_1^2 = 1 \)
- State \( |1\rangle \) has fiber of the \( G^+_3 \) elements \( so(\alpha, \beta, S)_{11} \) (1-type \( G^+_3 \) states):
  - \( \beta_3B_3 + \beta_2B_2 = (\beta_3 + \beta_2B_1)B_3, \beta_3^2 + \beta_2^2 = 1 \)

If \( B_2 \) is taken as complex plane, then
- State \( |0\rangle \) has fiber (level set) of the \( G^+_3 \) elements \( so(\alpha, \beta, S)_{00} \) (0-type \( G^+_3 \) states):
  - \( \alpha + \beta_2B_2, \alpha^2 + \beta_2^2 = 1 \)
- State \( |1\rangle \) has fiber of the \( G^+_3 \) elements \( so(\alpha, \beta, S)_{01} \) (1-type \( G^+_3 \) states):
  - \( \beta_1B_1 + \beta_3B_3 = (\beta_1 + \beta_3B_2)B_1, \beta_1^2 + \beta_3^2 = 1 \)

If \( B_3 \) is taken as complex plane, then
- State \( |0\rangle \) has fiber (level set) of the \( G^+_3 \) elements \( so(\alpha, \beta, S)_{00} \) (0-type \( G^+_3 \) states):
  - \( \alpha + \beta_3B_3, \alpha^2 + \beta_3^2 = 1 \)
- State \( |1\rangle \) has fiber of the \( G^+_3 \) elements \( so(\alpha, \beta, S)_{11} \) (1-type \( G^+_3 \) states):
  - \( \beta_1B_1 + \beta_2B_2 = (\beta_1 + \beta_2B_3)B_2, \beta_1^2 + \beta_2^2 = 1 \)

General definition of measurement in the suggested approach includes:

- the set of observables to be \( G^+_3 \).
- the set of states to be elements of \( G^+_3 \) (g-qubits up to some scalar factor).
- measurement of an observable
\[ C = C_0 + C_1B_1 + C_2B_2 + C_3B_3 \]

by g-qubit (wave function)
\[ \alpha + \beta_3B = \alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3 \]

is defined as
\[ (\alpha - \beta_3)C(\alpha + \beta_3) \]

with the result:
\[
C_0 + C_1B_1 + C_2B_2 + C_3B_3 \xrightarrow{\alpha + \beta_1\beta_2\beta_3 + \beta_2B_2 + \beta_3B_3} C_0 \\
+ (C_1[(\alpha^2 + \beta_1^2) - (\beta_2^2 + \beta_3^2)]) + 2C_2(\beta_1\beta_2 - \alpha\beta_3) + 2C_3(\alpha\beta_2 + \beta_1\beta_3)B_1 \\
+ (2C_4(\alpha\beta_3 + \beta_1\beta_2) + C_2[(\alpha^2 + \beta_2^2) - (\beta_1^2 + \beta_3^2)]) + 2C_3(\beta_2\beta_3 - \alpha\beta_1)B_2 \\
+ (2C_1(\beta_1\beta_3 - \alpha\beta_2) + 2C_2(\alpha\beta_1 + \beta_2\beta_3) + C_3[(\alpha^2 + \beta_3^2) - (\beta_1^2 + \beta_2^2)])B_3
\]

Probabilities of observed values are relative measures of the g-qubit fibers for each observable value received by the action of the states on observable.

The lift from \( C^2 \) to \( G^+_3 \) needs some \( \{B_1, B_2, B_3\} \) reference frame of unit value bivectors. This frame, as a solid, can be arbitrary rotated in three dimensions. In that sense we have principal fiber bundle \( G^+_3 \to C^2 \) with the standard fiber as group of rotations which is also effectively identified by elements of \( G^+_3 \).

Suppose we are interested in the probability of the result of measurement in which the observable component \( C_1B_1 \) does not change. This is relative measure of states \( \sqrt{\alpha^2 + \beta_1^2} \frac{\alpha}{\sqrt{\alpha^2 + \beta_1^2}} + \frac{\beta_1}{\sqrt{\alpha^2 + \beta_1^2}} B_1 \) in the measurements:

\[
\sqrt{\alpha^2 + \beta_1^2} \left( \frac{\alpha}{\sqrt{\alpha^2 + \beta_1^2}} - \frac{\beta_1}{\sqrt{\alpha^2 + \beta_1^2}} B_1 \right) C \sqrt{\alpha^2 + \beta_1^2} \left( \frac{\alpha}{\sqrt{\alpha^2 + \beta_1^2}} + \frac{\beta_1}{\sqrt{\alpha^2 + \beta_1^2}} B_1 \right)
\]

That measure is equal to \( \alpha^2 + \beta_1^2 \), that is equal to \( z_1^2 \) in the down mapping from \( G^+_3 \) to \( z_1|0\rangle + z_2 |1\rangle \).

Thus, we have clear explanation of common quantum mechanics wisdom on “probability of finding system in state \( |0\rangle \)”.

Similar calculations explain correspondence of \( \beta_1^2 + \beta_2^2 \) to \( z_1^2 \) in \( z_1|0\rangle + z_2 |1\rangle \) when the component \( C_1B_1 \) in measurement just got flipped.

Any arbitrary \( G^+_3 \) state \( s_0(\alpha, \beta, S) = \alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3 \) can be rewritten either as 0-type state or 1-type state:
\[
\alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3 = \alpha + l_s(\beta_1\beta_2\beta_3)\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}.
\]

where \( l_s(\beta_1\beta_2\beta_3) = \frac{\beta_1\beta_1 + \beta_2\beta_2 + \beta_3\beta_3}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}} \), 0-type.

or
\[
\alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3 = (\beta_3 + \beta_2B_1 - \beta_1B_2 - \alpha B_3)B_3 = (\beta_3 + l_s(\beta_2\beta_3\beta_3)\sqrt{\alpha^2 + \beta_1^2 + \beta_2^2})B_3.
\]
where \( I_5(\beta_2-\beta_1-a) = \frac{\beta_2B_1-\beta_1B_2-aB_3}{\sqrt{a^2+\beta_1^2+\beta_2^2}} \), 1-type.

All that means that any \( G_3^+ \) state \( \alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3 \) measuring arbitrary observable \( C_1B_1 + C_2B_2 + C_3B_3 \) does not change the observable projection onto plane of \( I_5(\beta_2,\beta_1,\beta_3) = \frac{\beta_3B_1+\beta_2B_2+\beta_1B_3}{\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}} \) and just flips the observable projection onto plane \( I_5(\beta_2-\beta_1-a) = \frac{\beta_3B_1-\beta_2B_2-\beta_1B_3}{\sqrt{a^2+\beta_1^2+\beta_2^2}} \).

3. Double Slit Experiment

Taking the set of g-qubits and projection of them onto \( C^2: \pi: G_3^+ \rightarrow C^2 \), we get fiber bundle. The projection depends on which basis bivector plane is selected as corresponding to formal imaginary unit plane. If we take, for example \( B_3 \), the projection is:

\[
\pi: so(\alpha, \beta, S) = \alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3 \rightarrow (\alpha + i\beta_3) + (\beta_2 + i\beta_1)
\]

Then for any \( z = (x_1+i\gamma_1, x_2+i\gamma_2) \in C^2 \) the fiber in \( G_3^+ \) consists of all elements \( F_z = x_1 + y_2B_1 + x_2B_2 + y_1B_3 \) with an arbitrary triple of orthonormal bivectors \( \{B_1, B_2, B_3\} \) satisfying multiplication rules. That particularly means that the standard fiber is group of rotations of basis bivectors in the standard fiber \( F_z \).

Thus, the fiber bundle is principal fiber bundle.

Let one first slit is only open, and the fiber, wave function, is some \( F^1 = x_1 + y_1B_1 + x_2B_2 + y_1B_3 \).

For the only open second slit the fiber is different: \( F^2 = x_1 + y_2B_1 + x_2B_2 + y_2B_3 \). When both slits are open the corresponding fiber is defined by connection, parallel transport anywhere between fibers \( F^1 \) and \( F^2 \).

Let we have a smooth curve \( \gamma(t, P_1, P_2), 0 \leq t \leq 1, \) connecting points \( P_1 = (x_1, y_1, x_2, y_1) \) and \( P_2 = (x_2, y_2, x_2, y_2) \), on three-dimensional sphere \( S^3 \) such that \( \gamma(0, P_1, P_2) = P_1 \) and \( \gamma(1, P_1, P_2) = P_2 \). The easiest way to define parallel transport is \( \gamma(t, P_1, P_2) = (1-t)P_1 + tP_2 \).

For convenience purposes let us write \( F^1 \) and \( F^2 \) as exponents:

\[
F^1 = x_1 + y_1B_1 + x_2B_2 + y_1B_3 = x_1 + \left(\frac{y_1}{\sqrt{(y_2)^2+(x_2)^2+(y_1)^2}}B_1 + \frac{x_1}{\sqrt{(y_2)^2+(x_2)^2+(y_1)^2}}B_2 + \frac{y_1}{\sqrt{(y_2)^2+(x_2)^2+(y_1)^2}}B_3\right)
\]

where \( \varphi_1 = \cos^{-1} x_1 \).

\[
I_{y_1} = \frac{y_1}{\sqrt{(y_2)^2+(x_2)^2+(y_1)^2}}B_1 + \frac{x_1}{\sqrt{(y_2)^2+(x_2)^2+(y_1)^2}}B_2 + \frac{y_1}{\sqrt{(y_2)^2+(x_2)^2+(y_1)^2}}B_3.
\]

Angle \( \varphi_1 \) is not uniquely defined since it can be any of \( \cos^{-1} x_1 \pm 2\pi k_1, \ k_1 = 0,1,2,... \), where \( \cos^{-1} x_1 \) is, by definition, taken from interval \([0,\pi]\). The angle \( \cos^{-1} x_1 \) will be denoted as \( \varphi_1(0) \).
\[ F^2 = x_1^2 + y_2^2B_1 + x_2^2B_2 + y_1^2B_3 = x_1^2 + \sqrt{(y_2^2)^2 + (x_2^2)^2 + (y_1^2)^2} \left( \frac{y_2^2}{\sqrt{(y_2^2)^2 + (x_2^2)^2 + (y_1^2)^2}}B_1 + \right. \\
\left. \frac{x_2^2}{\sqrt{(y_2^2)^2 + (x_2^2)^2 + (y_1^2)^2}}B_2 + \frac{y_1^2}{\sqrt{(y_2^2)^2 + (x_2^2)^2 + (y_1^2)^2}}B_3 \right) = e^{iS_2\varphi_2}, \]

where \( \varphi_2 = \cos^{-1} x_1^2 \),

\[ I_{S_2} = \frac{y_2^2}{\sqrt{(y_2^2)^2 + (x_2^2)^2 + (y_1^2)^2}}B_1 + \frac{x_2^2}{\sqrt{(y_2^2)^2 + (x_2^2)^2 + (y_1^2)^2}}B_2 + \frac{y_1^2}{\sqrt{(y_2^2)^2 + (x_2^2)^2 + (y_1^2)^2}}B_3. \]

As above, \( \varphi_2 = \cos^{-1} x_1^2 \pm 2\pi k_2, k_2 = 0,1,2, \ldots \) The angle \( \cos^{-1} x_1^2 \) will be denoted as \( \varphi_2(0) \).

Measurement of an observable

\[ C = C_0 + C_1B_1 + C_2B_2 + C_3B_3 = |C| \left( \frac{C_0}{|C|} + \frac{C_1}{|C|}B_1 + \frac{C_2}{|C|}B_2 + \frac{C_3}{|C|}B_3 \right) = \\
\left( \frac{C_0}{|C|} + \frac{1}{|C|^2} \left( \frac{C_1}{|C|} \frac{1}{\sqrt{1 - \frac{c_0^2}{|C|^2}}} - \frac{C_2}{|C|} \frac{1}{\sqrt{1 - \frac{c_2^2}{|C|^2}}} - \frac{C_3}{|C|} \frac{1}{\sqrt{1 - \frac{c_3^2}{|C|^2}}} \right) \right) = |C|e^{i\varphi}, \]

where \( |C| = \sqrt{C_0^2 + C_1^2 + C_2^2 + C_3^2} \), \( \varphi = \cos^{-1} \frac{C_0}{|C|} \),

\[ I_{S_1} = \frac{C_1}{|C|}B_1 + \frac{C_2}{|C|}B_2 + \frac{C_3}{|C|}B_3, \]

by the wave function \( e^{iS_1\varphi_1} \) is:

\[ M_1 = e^{-iS_1\varphi_1} |C|e^{i\varphi} e^{iS_1\varphi_1} \]

Measurement by \( e^{iS_2\varphi_2} \) is:

\[ M_2 = e^{-iS_2\varphi_2} |C|e^{i\varphi} e^{iS_2\varphi_2} \]

Measurement by any intermediate parallel transport wave function \((1 - t)e^{iS_1\varphi_1} + te^{-iS_2\varphi_2}\) then reads:

\[ (1 - t)^2e^{-iS_1\varphi_1}|C|e^{i\varphi} e^{iS_1\varphi_1} + t^2e^{-iS_2\varphi_2}|C|e^{i\varphi} e^{iS_2\varphi_2} + \]

\[ |C|(1 - t)(e^{-iS_1\varphi_1}e^{i\varphi} e^{iS_2\varphi_2} + e^{-iS_2\varphi_2} e^{i\varphi} e^{iS_1\varphi_1}) = \\
(1 - t)^2e^{-iS_1\varphi_1}|C|e^{i\varphi} e^{iS_1\varphi_1} + t^2e^{-iS_2\varphi_2}|C|e^{i\varphi} e^{iS_2\varphi_2} + \]

\[ t(1 - t)(e^{-iS_1\varphi_1}e^{iS_2\varphi_2} e^{i\varphi} e^{iS_2\varphi_2} + e^{-iS_2\varphi_2} e^{iS_1\varphi_1} e^{i\varphi} e^{iS_1\varphi_1}) = \\
(1 - t)^2M_1 + t^2M_2 + t(1 - t)(e^{-iS_2\varphi_2} e^{iS_1\varphi_1} M_1 + e^{-iS_1\varphi_1} e^{iS_2\varphi_2} M_2) \]

Let us make natural for double split experiment assumption \( S_1 = S_2 = S_0 \) (that is the two wave functions, measuring states, are of 0-type with identical bivector planes.) Then we get the measurement result by the intermediate parallel transport wave function:

\[ (1 - t)^2M_1 + t^2M_2 + t(1 - t)(e^{-iS_2\varphi_2} e^{iS_1\varphi_1} M_1 + e^{-iS_1\varphi_1} e^{iS_2\varphi_2} M_2) = \\
(1 - t)^2M_1 + t^2M_2 + t(1 - t)(e^{iS_0(\varphi_1 - \varphi_2)} M_1 + e^{iS_0(\varphi_2 - \varphi_1)} M_2) \]

It is easily seen that the result of measurement is \( M_1 \) when \( t = 0 \) and \( M_2 \) when \( t = 1 \).

Consider the following simplified scenario.
Assume we are only interested in the projections of $M_1$ and $M_2$ onto the plane of their rotations, $S_0$, $M_1(S_0)$ and $M_2(S_0)$. Then from the general formula

$$e^{iS_0}\varphi_2 e^{-i\varphi_1} = \cos\varphi_1 \cos\varphi_2 - (s_1 \cdot s_2) \sin\varphi_1 \sin\varphi_2 + i_3 s_2 \cos\varphi_1 \sin\varphi_2 + i_3 s_1 \cos\varphi_2 \sin\varphi_1$$

we get that up to some factors $e^{iS_0(\varphi_1-\varphi_2)} M_1(S_0)$ is $M_1(S_0)$ rotated in $S_0$ by angle $\varphi_1 - \varphi_2$ and $e^{iS_0(\varphi_2-\varphi_1)} M_2(S_0)$ is $M_2(S_0)$ rotated in $S_0$ by angle $\varphi_2 - \varphi_1$.

Without loss of generality suppose that the angles $\varphi_1(0)$ and $\varphi_2(0)$ are equal by values but opposite in sign:

$$\varphi_1(0) = -\varphi_0, \varphi_2(0) = \varphi_0,$$

$$\varphi_1(0) - \varphi_2(0) = -2\varphi_0$$

$$\varphi_2(0) - \varphi_1(0) = 2\varphi_0$$

Then it follows that in Clifford translations the projection $M_1(S_0)$ rotates in $S_0$ additionally by $-2(\varphi_0 \pm \pi(k_1 - k_2))$, $k_1 = 0,1,2,...$, $k_2 = 0,1,2,...$, and projection $M_2(S_0)$ rotates in $S_0$ additionally by $2(\varphi_0 \pm \pi(k_1 - k_2))$, $k_1 = 0,1,2,...$, $k_2 = 0,1,2,...$.

Thus, we get infinite number of copies of $M_1(S_0)$ and $M_2(S_0)$ with varying values depending every time on uniformly distributed, by assumption, value of $t$, $0 \leq t \leq 1$, and separated by $\pm\pi$ along the big circle of intersection of plane $S_0$ with the sphere $S^3$.

4. Conclusions

It was demonstrated that the geometric algebra formalism along with generalization of complex numbers and subsequent lift of the two-dimensional Hilbert space valued qubits to geometrically feasible elements of even subalgebra of geometric algebra in three dimensions allows, particularly, to resolve the double split experiment results with diffraction patterns inherent to wave diffraction. This weirdness of the double split experiment is milestone of all further difficulties in interpretation of conventional quantum mechanics.

References


