Explaining Double Split Experiment with Geometrical Algebra Formalism

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Abstract: The Geometric Algebra formalism opens the door to developing a theory upgrading conventional quantum mechanics. Generalizations, stemming from implementation of complex numbers as geometrically feasible objects in three dimensions; unambiguous definition of states, observables, measurements bring into reality clear explanations of conventional weird quantum mechanical features, particularly the results of double split experiments where particles create diffraction patterns inherent to wave diffraction. This weirdness of the double split experiment is milestone of all further difficulties in interpretation of quantum mechanics.

1. Introduction. Working with g-qubits instead of qubits.

Theory of upgrading conventional quantum mechanics has been under development [1], [2] [3], [4].

The main novel features are:

- Replacing complex numbers with elements of even subalgebra of geometric algebra in three dimensions, that's by elements of the form "scalar plus bivector".
- The objects identifying physical media are of the same structure: explicitly defined plane along with angle of rotation in that plane.
- Operators acting on the objects are operators of rotation having the same structure: scalar plus bivector. That is the measurement operation.
- Mapping of operator to the result of measurement, that is collapse, creates a "particle".
- The operators can be identified by points on the three-sphere $\mathbb{S}^3$ and are connected, due to hedgehog theorem, by Clifford translations. Modifying the operators due to Clifford translation is identified by the generalization of Schrodinger equation containing unit bivectors in three dimensions instead of formal imaginary unit

Qubits, identifying states in conventional quantum mechanics, mathematically are elements of two-dimensional complex spaces:

$\begin{pmatrix} x_1+iy_1 \\ x_2+iy_2 \end{pmatrix}$, $\|x_1+iy_1\|^2 + \|x_2+iy_2\|^2 = 1$, unit value element of C^2.

Imaginary unit i is used formally, without geometrical identification in three dimensions. In another accepted notations a qubit is:

$$C^2 \ni \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = z_1|0\rangle + z_2|1\rangle$$

In the suggested formalism complex numbers $x+iy$ are replaced with elements of G_3^+, subalgebra of G_3.

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The G_3 elements of the form $M_3 = \alpha + I_S\beta$, where I_S is some unit bivector arbitrary placed in three-dimensional space, comprise so called *even* subalgebra of algebra G_3.This subalgebra is denoted by G_3^+, [3], [4]. Elements of G_3^+ can be depict as in Fig 1.1.

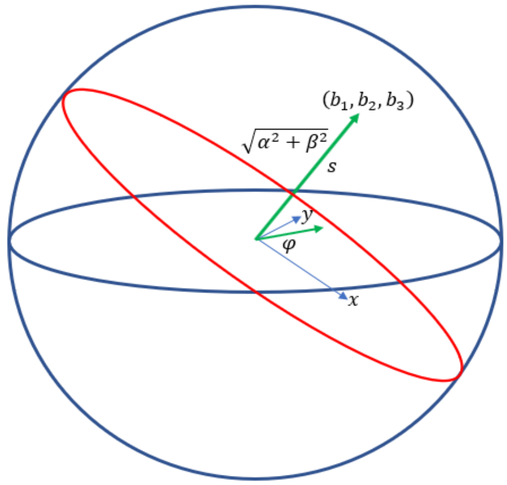

Fig.1.1. An element of G_3^+

Unit value elements of G_3^+, $\alpha^2 + \beta^2 = 1$, will be called *g-qubits.* The wave functions (states) implemented as g-qubits store much more information than qubits, see Fig 1.2.

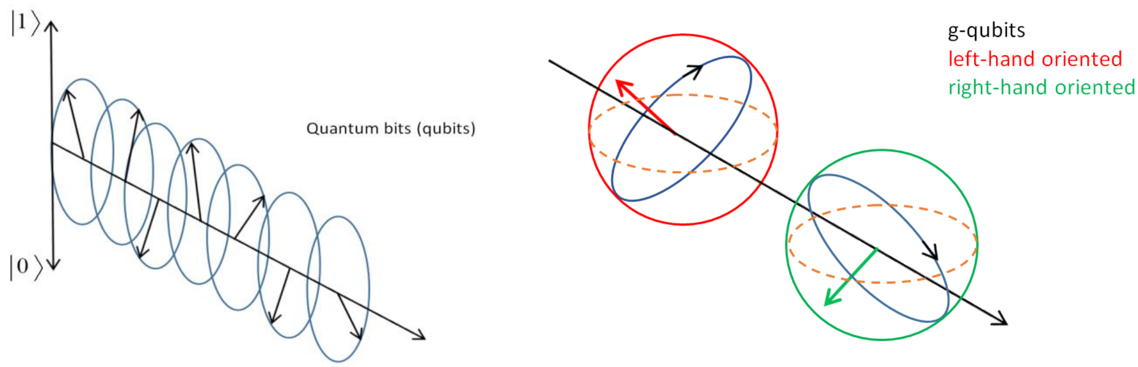

Fig. 1.2. Geomectrically picted qubits and g-qubits

2. Lift of qubits to g-qubits, fiber bundles and probabilities

Take right-hand screw oriented basis $\{B_1, B_2, B_3\}$ of unit value bivectors, with the multiplication rules $B_1B_2 = -B_3$, $B_1B_3 = B_2$, $B_2B_3 = -B_1$, $I_3B_1I_3B_2I_3B_3 = I_3$ (or equivalently $B_1B_2B_3 = 1$), where I_3 is oriented unit value volume in three dimensions named also pseudoscalar.

Quantum mechanical qubit state, $|\psi\rangle = z_1|0\rangle + z_2|1\rangle$, is linear combination of two basis states $|0\rangle$ and $|1\rangle$. In the G_3^+ terms these two states correspond to the following classes of equivalence in G_3^+, depending on which basis bivector is selected as complex plane:

If B_1 is taken as complex plane, then

- State $|0\rangle$ has fiber (level set) of the G_3^+ elements $so(\alpha, \beta, S)_{|0\rangle}$ (0-type G_3^+ states):

  - $\alpha + \beta_1 B_1,\ \alpha^2 + \beta_1^2 = 1$

- State $|1\rangle$ has fiber of the G_3^+ elements $so(\alpha, \beta, S)_{|1\rangle}$ (1-type G_3^+ states):

  - $\beta_3\ B_3 + \beta_2\ B_2 = (\beta_3 + \beta_2 B_1)B_3,\ \beta_3^2 + \beta_2^2 = 1$

If B_2 is taken as complex plane, then

- State $|0\rangle$ has fiber (level set) of the G_3^+ elements $so(\alpha, \beta, S)_{|0\rangle}$ (0-type G_3^+ states):

  - $\alpha + \beta_2 B_2,\ \alpha^2 + \beta_2^2 = 1$

- State $|1\rangle$ has fiber of the G_3^+ elements $so(\alpha, \beta, S)_{|1\rangle}$ (1-type G_3^+ states):

  - $\beta_1\ B_1 + \beta_3\ B_3 = (\beta_1 + \beta_3 B_2)B_1,\ \beta_1^2 + \beta_3^2 = 1$

If B_3 is taken as complex plane, then

- State $|0\rangle$ has fiber (level set) of the G_3^+ elements $so(\alpha, \beta, S)_{|0\rangle}$ (0-type G_3^+ states):

  - $\alpha + \beta_3 B_3,\ \alpha^2 + \beta_3^2 = 1$

- State $|1\rangle$ has fiber of the G_3^+ elements $so(\alpha, \beta, S)_{|1\rangle}$ (1-type G_3^+ states):

  - $\beta_1\ B_1 + \beta_2\ B_2 = (\beta_2 + \beta_1 B_3)B_2,\ \beta_2^2 + \beta_1^2 = 1$

General definition of measurement in the suggested approach includes:

- the set of observables to be G_3^+,

- the set of states to be elements of G_3^+ (g-qubits up to some scalar factor),

- measurement of an observable

$$C = C_0 + C_1 B_1 + C_2 B_2 + C_3 B_3$$

by g-qubit (wave function)

$$\alpha + I_S \beta = \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$$

is defined as

$$(\alpha - I_S\beta)C(\alpha + I_S\beta)$$

with the result:

$$C_0 + C_1B_1 + C_2B_2 + C_3B_3 \xrightarrow{\alpha+\beta_1B_1+\beta_2B_2+\beta_3B_3} C_0 + \big(C_1[(\alpha^2 + \beta_1^2) - (\beta_2^2 + \beta_3^2)] + 2C_2(\beta_1\beta_2 - \alpha\beta_3) + 2C_3(\alpha\beta_2 + \beta_1\beta_3)\big)B_1 + \big(2C_1(\alpha\beta_3 + \beta_1\beta_2) + C_2[(\alpha^2 + \beta_2^2) - (\beta_1^2 + \beta_3^2)] + 2C_3(\beta_2\beta_3 - \alpha\beta_1)\big)B_2 + (2C_1(\beta_1\beta_3 - \alpha\beta_2) + 2C_2(\alpha\beta_1 + \beta_2\beta_3) + C_3[(\alpha^2 + \beta_3^2) - (\beta_1^2 + \beta_2^2)])B_3$$

Probabilities of observed values are relative measures of the g-qubit fibers for each observable value received by the action of the states on observable.

The lift from C^2 to G_3^+ needs some $\{B_1, B_2, B_3\}$ reference frame of unit value bivectors. This frame, as a solid, can be arbitrary rotated in three dimensions. In that sense we have principal fiber bundle $G_3^+ \to C^2$ with the standard fiber as group of rotations which is also effectively identified by elements of G_3^+.

Suppose we are interested in the probability of the result of measurement in which the observable component C_1B_1 does not change. This is relative measure of states $\sqrt{\alpha^2 + \beta_1^2}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta_1^2}} + \frac{\beta_1}{\sqrt{\alpha^2+\beta_1^2}}B_1\right)$ in the measurements:

$$\sqrt{\alpha^2 + \beta_1^2}\left(\frac{\alpha}{\sqrt{\alpha^2 + \beta_1^2}} - \frac{\beta_1}{\sqrt{\alpha^2 + \beta_1^2}}\ B_1\right) C\sqrt{\alpha^2 + \beta_1^2}\left(\frac{\alpha}{\sqrt{\alpha^2 + \beta_1^2}} + \frac{\beta_1}{\sqrt{\alpha^2 + \beta_1^2}}B_1\right)$$

That measure is equal to $\alpha^2 + \beta_1^2$, that is equal to z_1^2 in the down mapping from G_3^+ to $z_1|0\rangle + z_2\ |1\rangle$. Thus, we have clear explanation of common quantum mechanics wisdom on "probability of finding system in state $|0\rangle$".

Similar calculations explain correspondence of $\beta_3^2 + \beta_2^2$ to z_2^2 in $z_1|0\rangle + z_2\ |1\rangle$ when the component C_1B_1 in measurement just got flipped.

Any arbitrary G_3^+ state $so(\alpha, \beta, S) = \alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3$ can be rewritten either as 0-type state or 1-type state:

$\alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3 = \alpha + I_{S(\beta_1,\beta_2,\beta_3)}\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}$,

where $I_{S(\beta_1,\beta_2,\beta_3)} = \frac{\beta_1B_1 + \beta_2B_2 + \beta_3B_3}{\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}}$, 0-type,

or

$\alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3 = (\beta_3 + \beta_2B_1 - \beta_1B_2 - \alpha B_3)B_3 = \left(\beta_3 + I_{S(\beta_2,-\beta_1,-\alpha)}\sqrt{\alpha^2 + \beta_1^2 + \beta_2^2}\right)B_3$,

where $I_{S(\beta_2,-\beta_1,-\alpha)} = \frac{\beta_2B_1 - \beta_1B_2 - \alpha B_3}{\sqrt{\alpha^2+\beta_1^2+\beta_2^2}}$, 1-type.

All that means that any G_3^+ state $\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$ measuring arbitrary observable $C_1 B_1 + C_2 B_2 + C_3 B_3$ does not change the observable projection onto plane of $I_{S(\beta_1,\beta_2,\beta_3)} = \frac{\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3}{\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}}$ and just flips the observable projection onto plane $I_{S(\beta_2,-\beta_1,-\alpha)} = \frac{\beta_2 B_1 - \beta_1 B_2 - \alpha B_3}{\sqrt{\alpha^2+\beta_1^2+\beta_2^2}}$.

3. Double slit experiment

Taking the set of g-qubits and projection of them onto C^2: $\pi: G_3^+ \to C^2$, we get fiber bundle. The projection depends on which basis bivector plane is selected as corresponding to formal imaginary unit plane. If we take, for example B_3, the projection is:

$$\pi: so(\alpha,\beta,S) = \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 \to \begin{pmatrix} \alpha + i\beta_3 \\ \beta_2 + i\beta_1 \end{pmatrix}$$

Then for any $z = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \in C^2$ the fiber in G_3^+ consists of all elements $F_z = x_1 + y_2 B_1 + x_2 B_2 + y_1 B_3$ with an arbitrary triple of orthonormal bivectors $\{B_1, B_2, B_3\}$ satisfying multiplication rules. That particularly means that the standard fiber is group of rotations of basis bivectors in the standard fiber F_z. Thus, the fiber bundle is principal fiber bundle.

Let one first slit is only open, and the fiber, wave function, is some $F^1 = x_1^1 + y_2^1 B_1 + x_2^1 B_2 + y_1^1 B_3$. For the only open second slit the fiber is different: $F^2 = x_1^2 + y_2^2 B_1 + x_2^2 B_2 + y_1^2 B_3$. When both slits are open the corresponding fiber is defined by connection, parallel transport anywhere between fibers F^1 and F^2.

Let we have a smooth curve $\gamma(t, P_1, P_2)$, $0 \le t \le 1$, connecting points $P_1 = (x_1^1, y_2^1, x_2^1, y_1^1)$ and $P_2 = (x_1^2, y_2^2, x_2^2, y_1^2)$, on three-dimensional sphere $\mathbb{S}^3$ such that $\gamma(0, P_1, P_2) = P_1$ and $\gamma(1, P_1, P_2) = P_2$. The easiest way to define parallel transport is $\gamma(t, P_1, P_2) = (1-t)P_1 + tP_2$.

For convenience purposes let us write F^1 and F^2 as exponents:

$$F^1 = x_1^1 + y_2^1 B_1 + x_2^1 B_2 + y_1^1 B_3 = x_1^1 + \sqrt{(y_2^1)^2 + (x_2^1)^2 + (y_1^1)^2} \left(\frac{y_2^1}{\sqrt{\left(y_2^1\right)^2 + \left(x_2^1\right)^2 + \left(y_1^1\right)^2}} B_1 + \frac{x_2^1}{\sqrt{\left(y_2^1\right)^2 + \left(x_2^1\right)^2 + \left(y_1^1\right)^2}} B_2 + \frac{y_1^1}{\sqrt{\left(y_2^1\right)^2 + \left(x_2^1\right)^2 + \left(y_1^1\right)^2}} B_3 \right) = e^{I_{S_1}\varphi_1},$$

where $\varphi_1 = cos^{-1} x_1^1$,

$$I_{S_1} = \frac{y_2^1}{\sqrt{\left(y_2^1\right)^2 + \left(x_2^1\right)^2 + \left(y_1^1\right)^2}} B_1 + \frac{x_2^1}{\sqrt{\left(y_2^1\right)^2 + \left(x_2^1\right)^2 + \left(y_1^1\right)^2}} B_2 + \frac{y_1^1}{\sqrt{\left(y_2^1\right)^2 + \left(x_2^1\right)^2 + \left(y_1^1\right)^2}} B_3.$$

Angle φ_1 is not uniquely defined since it can be any of $\cos^{-1} x_1^1 \pm 2\pi k_1$, $k_1 = 0,1,2, \dots$, where $\cos^{-1} x_1^1$ is, by definition, taken from interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The angle $\cos^{-1} x_1^1$ will be denoted as $\varphi_1(0)$.

$$F^2 = x_1^2 + y_2^2 B_1 + x_2^2 B_2 + y_1^2 B_3 = x_1^2 + \sqrt{(y_2^2)^2 + (x_2^2)^2 + (y_1^2)^2}\left(\frac{y_2^2}{\sqrt{\left(y_2^2\right)^2+\left(x_2^2\right)^2+\left(y_1^2\right)^2}} B_1 + \frac{x_2^2}{\sqrt{\left(y_2^2\right)^2+\left(x_2^2\right)^2+\left(y_1^2\right)^2}} B_2 + \frac{y_1^2}{\sqrt{\left(y_2^2\right)^2+\left(x_2^2\right)^2+\left(y_1^2\right)^2}} B_3\right) = e^{I_{S_2}\varphi_2},$$

where $\varphi_2 = cos^{-1} x_1^2$,

$$I_{S_2} = \frac{y_2^2}{\sqrt{\left(y_2^2\right)^2+\left(x_2^2\right)^2+\left(y_1^2\right)^2}} B_1 + \frac{x_2^2}{\sqrt{\left(y_2^2\right)^2+\left(x_2^2\right)^2+\left(y_1^2\right)^2}} B_2 + \frac{y_1^2}{\sqrt{\left(y_2^2\right)^2+\left(x_2^2\right)^2+\left(y_1^2\right)^2}} B_3.$$

As above, $\varphi_2 = \cos^{-1} x_1^2 \pm 2\pi\, k_2$, $k_2 = 0,1,2, \dots$ The angle $\cos^{-1} x_1^2$ will be denoted as $\varphi_2(0)$.

Measurement of an observable

$$C = C_0 + C_1 B_1 + C_2 B_2 + C_3 B_3 = |C|\left(\frac{C_0}{|C|} + \frac{C_1}{|C|} B_1 + \frac{C_2}{|C|} B_2 + \frac{C_3}{|C|} B_3\right) =$$

$$|C|\left(\frac{C_0}{|C|} + \sqrt{1 - \frac{C_0^2}{|C|^2}}\left(\frac{C_1}{|C|\sqrt{1-\frac{C_0^2}{|C|^2}}} B_1 + \frac{C_2}{|C|\sqrt{1-\frac{C_0^2}{|C|^2}}} B_2 + \frac{C_3}{|C|\sqrt{1-\frac{C_0^2}{|C|^2}}} B_3\right)\right) = |C| e^{I_S \varphi},$$

where $|C| = \sqrt{C_0^2 + C_1^2 + C_2^2 + C_3^2}$, $\varphi = \cos^{-1}\frac{C_0}{|C|}$, $I_S = \frac{C_1}{|C|\sqrt{1-\frac{C_0^2}{|C|^2}}} B_1 + \frac{C_2}{|C|\sqrt{1-\frac{C_0^2}{|C|^2}}} B_2 + \frac{C_3}{|C|\sqrt{1-\frac{C_0^2}{|C|^2}}} B_3$,

by the wave function $e^{I_{S_1}\varphi_1}$ is:

$$M_1 = e^{-I_{S_1}\varphi_1}\, |C| e^{I_S \varphi}\, e^{I_{S_1}\varphi_1}$$

Measurement by $e^{I_{S_2}\varphi_2}$ is:

$$M_2 = e^{-I_{S_2}\varphi_2}\, |C| e^{I_S \varphi} e^{I_{S_2}\varphi_2}$$

Measurement by any intermediate parallel transport wave function $(1-t)e^{I_{S_1}\varphi_1} + t e^{I_{S_2}\varphi_2}$ then reads:

$$(1-t)^2 e^{-I_{S_1}\varphi_1} |C| e^{I_S \varphi}\, e^{I_{S_1}\varphi_1} + t^2 e^{-I_{S_2}\varphi_2} |C| e^{I_S \varphi} e^{I_{S_2}\varphi_2} +$$

$$|C|t(1-t)(e^{-Is_1\varphi_1}e^{Is\varphi}e^{Is_2\varphi_2}+e^{-Is_2\varphi_2}e^{Is\varphi}e^{Is_1\varphi_1}) =$$

$$(1-t)^2 e^{-Is_1\varphi_1}|C|e^{Is\varphi}e^{Is_1\varphi_1}+t^2 e^{-Is_2\varphi_2}|C|e^{Is\varphi}e^{Is_2\varphi_2}+$$

$$t(1-t)(e^{-Is_1\varphi_1}e^{Is_2\varphi_2}e^{-Is_2\varphi_2}|C|e^{Is\varphi}e^{Is_2\varphi_2}+e^{-Is_2\varphi_2}e^{Is_1\varphi_1}e^{-Is_1\varphi_1}|C|e^{Is\varphi}e^{Is_1\varphi_1}) =$$

$$(1-t)^2 M_1+t^2 M_2+t(1-t)(e^{-Is_2\varphi_2}e^{Is_1\varphi_1}M_1+e^{-Is_1\varphi_1}e^{Is_2\varphi_2}M_2)$$

Let us make natural for double split experiment assumption $S_1=S_2=S_0$ (that is the two wave functions, measuring states, are of 0-type with identical bivector planes.) Then we get the measurement result by the intermediate parallel transport wave function:

$$(1-t)^2 M_1+t^2 M_2+t(1-t)(e^{-Is_2\varphi_2}e^{Is_1\varphi_1}M_1+e^{-Is_1\varphi_1}e^{Is_2\varphi_2}M_2) =$$

$$(1-t)^2 M_1+t^2 M_2+t(1-t)\left(e^{Is_0(\varphi_1-\varphi_2)}M_1+e^{Is_0(\varphi_2-\varphi_1)}M_2\right)$$

It is easily seen that the result of measurement is M_1 when $t=0$ and M_2 when $t=1$.

Consider the following simplified scenario.

Assume we are only interested in the projections of M_1 and M_2 onto the plane of their rotations, S_0, $M_1(S_0)$ and $M_2(S_0)$. Then from the general formula

$$e^{Is_2\varphi_2}e^{Is_1\varphi_1}=\cos\varphi_1\cos\varphi_2-(s_1\cdot s_2)\sin\varphi_1\sin\varphi_2+I_3 s_2\cos\varphi_1\sin\varphi_2+I_3 s_1\cos\varphi_2\sin\varphi_1 - I_3(s_2\times s_1)\sin\varphi_1\sin\varphi_2$$

we get that up to some factors $e^{Is_0(\varphi_1-\varphi_2)}M_1(S_0)$ is $M_1(S_0)$ rotated in S_0 by angle $\varphi_1-\varphi_2$ and $e^{Is_0(\varphi_2-\varphi_1)}M_2(S_0)$ is $M_2(S_0)$ rotated in S_0 by angle $\varphi_2-\varphi_1$.

Without loss of generality suppose that the angles $\varphi_1(0)$ and $\varphi_2(0)$are equal by values but opposite in sign:

$$\varphi_1(0)=-\varphi_0,\ \ \varphi_2(0)=\varphi_0,$$

$$\varphi_1(0)-\varphi_2(0)=-2\varphi_0$$

$$\varphi_2(0)-\varphi_1(0)=2\varphi_0$$

Then it follows that in Clifford translations the projection $M_1(S_0)$ rotates in S_0 additionally by $-2\left(\varphi_0\pm\pi(k_1-k_2)\right)$, $k_1=0,1,2,\dots$, $k_2=0,1,2,\dots$, and projection $M_2(S_0)$ rotates in S_0 additionally by $2\left(\varphi_0\pm\pi(k_1-k_2)\right)$, $k_1=0,1,2,\dots$, $k_2=0,1,2,\dots$.

Thus, we get infinite number of copies of $M_1(S_0)$ and $M_2(S_0)$ with varying values depending every time on uniformly distributed, by assumption, value of t, $0\le t\le 1$, and separated by $\pm\pi$ along the big circle of intersection of plane S_0 with the sphere $\mathbb{S}^3$.

4. Conclusions

It was demonstrated that the geometric algebra formalism along with generalization of complex numbers and subsequent lift of the two-dimensional Hilbert space valued qubits to geometrically feasible elements of even subalgebra of geometric algebra in three dimensions allows, particularly, to resolve the double split experiment results with diffraction patterns inherent to wave diffraction. This weirdness of the double split experiment is milestone of all further difficulties in interpretation of conventional quantum mechanics.

Works Cited

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