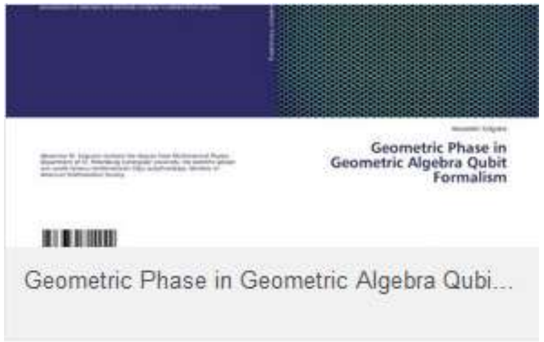
What this slideshow has inside

What follows is a brief introduction into innovative approach aimed at replacing commonly accepted the two-state, qubit, formalism of Quantum Mechanics. The final goal is implementation of a new mathematical tool to deal with Topological Quantum Computing.

With the generalization of complex numbers to elements of even subalgebra of the Clifford algebra over three dimensional space we not only get explicit, unambiguous, geometrically and physically clear interpretation but also a deeper theory than commonly accepted variant of quantum mechanics.

A lot of details can be found in the book:

Alexander Soiguine, Geometric Phase in Geometric Algebra Qubit Formalism, Saarbrücken: LAMBERT Academic Publishing, December 2015.

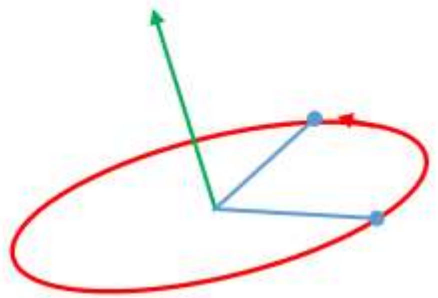


Let be honest with ourselves: quantum mechanics in its existing formulation is dead-end and actually is no-go-around obstacle for creating quantum computers. People responsible for that lived in the first quarter of the 20th century and knew a lot about matrix linear algebra. As usually happens, a tool selected for implementation of some idea becomes the selected option not because it is the best one but just because it is either easily available or the only one known.

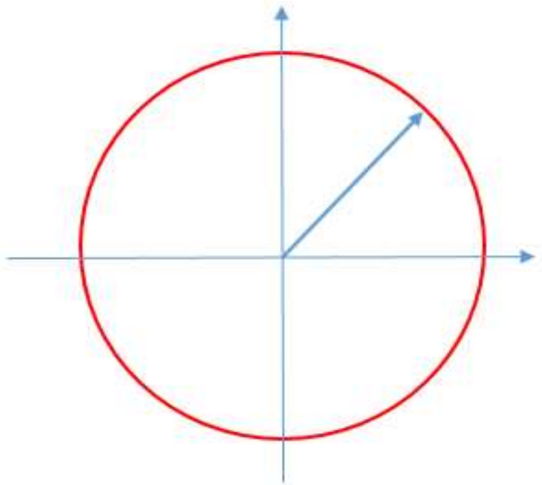
Luckily for them, but unfortunately for us, those people could successfully describe some experimental results using eigenvalues, eigenstates, self-adjoint operators, all that garbage not possessing understandable roots in the world of real physical observables, states, measurements.

Complex numbers and g-qubits

The very first critical thing for the whole approach is to generalize algebraically formal using of complex numbers to geometrically clear, unambiguous objects – elements of even subalgebra of geometric algebra over three dimensional space. Such objects are identified by an arbitrary oriented plane in three dimensions and angle of rotation in that plane. I will call such objects **g-qubits**, in the case when they have unit value, to distinguish them from qubits as complex valued two-dimensional unit value vectors.



g-qubit as oriented plane
in three dimensions together
with angle of rotation in
that plane



complex number as vector
rotated by an angle in
unspecified plane

Complex numbers of unit value, when taken as operators (“states” as will be seen later), can act, through multiplication, on other complex numbers (“observables”) causing them get rotated:

$$S = \alpha + i\beta = \left(\alpha^2 + \beta^2 = 1\right) \equiv \cos\varphi + i\sin\varphi$$

$$\left(operator, "state"\right)$$

$$O = x + iy = r\left(\frac{x}{r} + i\frac{y}{r}\right) = \left(x^2 + y^2 = r^2\right) \equiv r\left(\cos\psi + i\sin\psi\right)$$

$$\left(operand, "observable"\right)$$

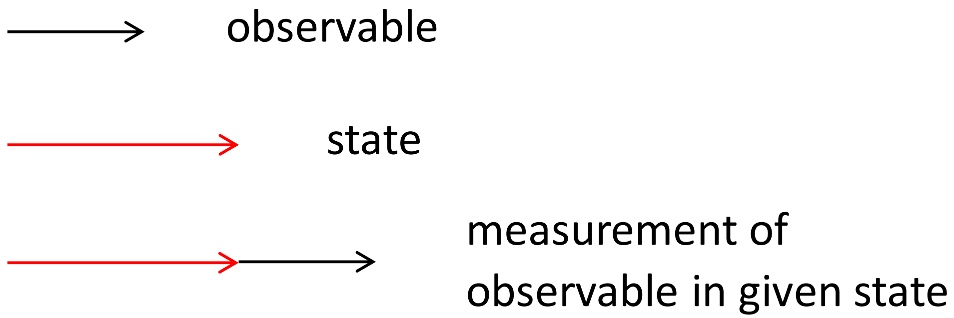
Then watching the result of measurement is:

$$SO = r\left(\cos\varphi\cos\psi - \sin\varphi\sin\psi + i\left(\sin\varphi\cos\psi + \cos\varphi\sin\psi\right)\right) =$$
$$r\left(\cos\left(\varphi + \psi\right) + i\sin\left(\varphi + \psi\right)\right)$$

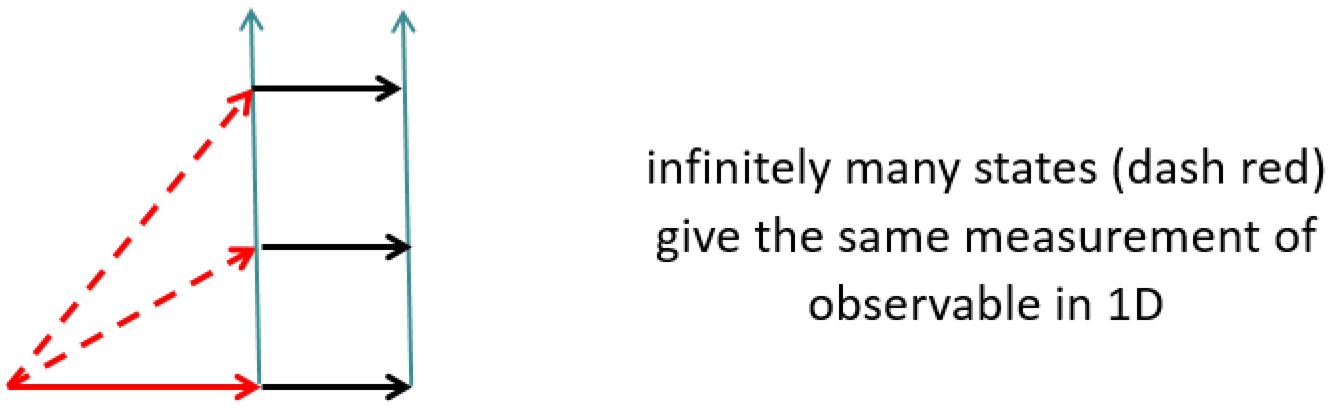
Obviously, the above multiplication is commutative

Operators, “states”, and observables are there mathematical objects of the same nature, together with the result of action of the operator on observable (“measurement”). Due to the way a complex number as operator transforms observable, it does not matter would we track the operator evaluating in time or the result of its action on observable. But … not rigorous distinguishing between operators and operands is maybe the main reason of ambiguous terminology in quantum mechanics (“find observable in state …” – what the hell is that?)

By the way, similar situation with states and observables takes place, for example, in the classical mechanics massive point movement dynamics. Take the simplest example: the point is watched as moving along straight line



If we continue watching on a straight line but actual process takes place in two dimensions we get



This elementary example is ideologically very important for further considerations.

When dealing with g-qubits as operators acting on observables (operands) the situation becomes critically different because the plane of rotation becomes clearly defined in three dimensions, not like some tacit not explicitly declared plane as in the case of complex numbers, and that plane of rotation not necessarily coincides with the plane of observable, the observable is also an object in three dimensions of the same nature as g-qubit (state). Measurement, watching the result of action of operator, state, g-qubit, differs from simple commutative multiplication of complex numbers.

When an observable is element of even subalgebra of geometric algebra over three dimensions, the subalgebra will be denoted as G_3^+, it has the form

$$o(\alpha, \beta, S) = \alpha + I_S \beta$$

where α and β are scalars and I_S is unit value bivector in a plane S belonging to three dimensional space. The above sum is not a sum of similar geometrical objects with the result of the same nature. It is the result of putting two objects of different nature into “one bag”. Scalars α and β keep information about angle and I_S defines a plane in three dimensions. In other words $\alpha + I_S \beta$ is something defining rotation by given angle in a given plane.

The elements of the subalgebra playing the role of operators (states) only differ from observables by the requirement of unit value:

$$s(\alpha,\beta,S)=\alpha+I_S\beta,\alpha^2+\beta^2=1$$

Action of states on observables is defined as

$$o(\alpha_1,\beta_1,S_1)\xrightarrow{s(\alpha_2,\beta_2,S_2)}(\alpha_2-I_{S_2}\beta_2)(\alpha_1+I_{S_1}\beta_1)(\alpha_2+I_{S_2}\beta_2)\equiv$$

$$\overline{s(\alpha_2,\beta_2,S_2)}o(\alpha_1,\beta_1,S_1)s(\alpha_2,\beta_2,S_2)$$

The line above the first term means “complex conjugation” (see the work: A. Soiguine, "Complex Conjugation - Relative to What?," in *Clifford Algebras with Numeric and Symbolic Computations*, Boston, Birkhauser, 1996, pp. 285-294.)

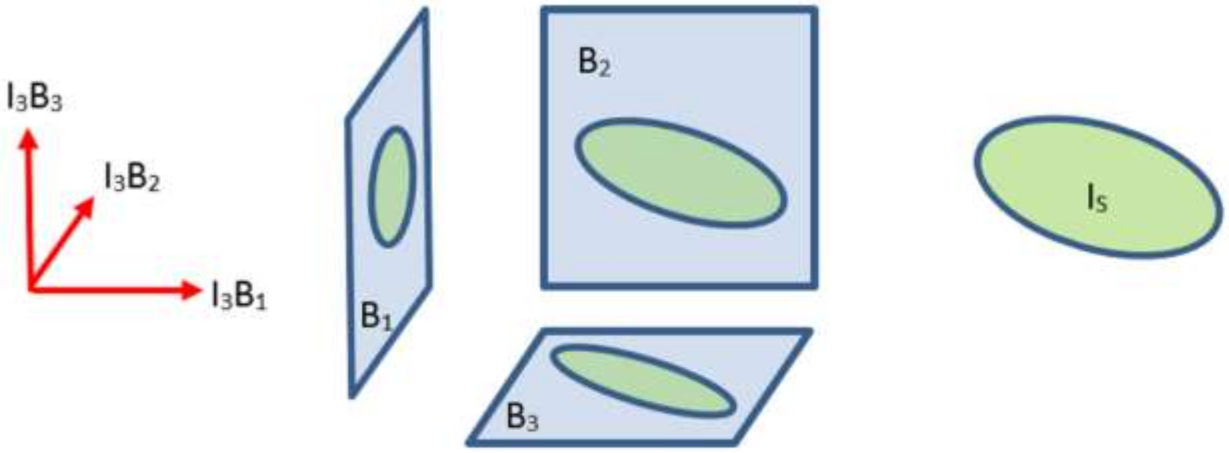
The elements I_S have the same property as “imaginary unit”: $(I_S)^2=-1$

The action of operators (states) in G_3^+ becomes identical to complex number rotations if only the planes S_1 and S_2 are identical. In that case that common plane plays the role of traditional “complex plane” and everything is going on in two dimensions not in three dimensional world.

The operators, states, as the G_3^+ objects, have unit value in the sense of the norm:

$$\left\| s(\alpha_2, \beta_2, S_2) \right\|^2 = \overline{s(\alpha_2, \beta_2, S_2)} s(\alpha_2, \beta_2, S_2) =$$
$$(\alpha_2 - I_{S_2}\beta_2)(\alpha_2 + I_{S_2}\beta_2) = \alpha_2^2 + \beta_2^2 = 1$$

We are very close to explaining relations between usual quantum mechanical qubits as complex two-dimensional vectors and g-qubits, states from G_3^+ . Before doing that it is necessary to show how to represent states when G_3^+ has a basis, three basis unit value bivectors to expand in them any bivector:

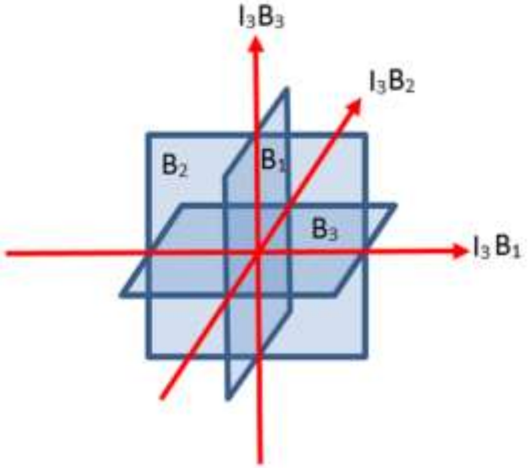


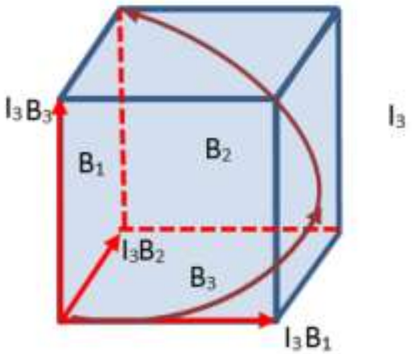
The red arrow coordinate system on the left side consists of three vectors dual to basis bivectors.

We need an agreement about the three dimension coordinate system orientation. The two options are: left-hand or right-hand orientation. I am choosing the right-hand. I think it is easier imaginable for everyone – the most of people have had experience of opening a wine bottle with a corkscrew, and I've never seen the left-hand oriented corkscrew. Though, I knew a guy, left-handed guy, and it was a tough problem for him to imagine the right-hand screw directional movement associated with clockwise or counterclockwise rotations.

In the above picture right-handed orientation means that when you rotate vector I_3B_1 to I_3B_2 by angle less than π you get movement of the screw in the direction of I_3B_3.

An explanation is necessary why do I call the items $I_3 B_i$ vectors and what is I_3. The last one is oriented volume in three dimensions particularly implementing duality between vectors and bivectors that is seen from the picture:





The basis bivectors satisfy, with their right-hand screw orientation agreement, multiplication rules:

$$B_1 B_2 = -B_3 \qquad B_1 B_3 = B_2 \qquad B_2 B_3 = -B_1$$

In a given basis of bivectors any bivector is expandable and an element from G_3^+ takes the form:

$$\alpha + I_S \beta = \alpha + \beta\left(b_1 B_1 + b_2 B_2 + b_3 B_3\right) \equiv \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$$

Now we can show how to establish correspondence between geometric algebra G_3^+ states and conventional quantum mechanical states as complex two dimensional vectors:

$$|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, z_1^2 + z_2^2 = z_1 \tilde{z}_1 + z_2 \tilde{z}_2 = 1, z_k = z_k^1 + i z_k^2, k = 1,2$$

We can arrange the mapping in at least three ways:

$$\left.\begin{array}{l}\alpha+\beta_1B_1+\beta_2B_2+\beta_3B_3=\\ \alpha+\beta_1B_1+(\beta_3+\beta_2B_1)B_3=\\ \alpha+\beta_2B_2+(\beta_1+\beta_3B_2)B_1=\\ \alpha+\beta_3B_3+(\beta_2+\beta_1B_3)B_2\end{array}\right\}\Rightarrow\begin{cases}\begin{pmatrix}\alpha+i\beta_1\\ \beta_3+i\beta_2\end{pmatrix}, i=B_1; B_2, B_3 unspecified\\ \begin{pmatrix}\alpha+i\beta_2\\ \beta_1+i\beta_3\end{pmatrix}, i=B_2; B_1, B_3 unspecified\\ \begin{pmatrix}\alpha+i\beta_3\\ \beta_2+i\beta_1\end{pmatrix}, i=B_3; B_1, B_2 unspecified\end{cases}$$

$$=\begin{cases}(\alpha+i\beta_1)\begin{pmatrix}1\\0\end{pmatrix}+(\beta_3+i\beta_2)\begin{pmatrix}0\\1\end{pmatrix}, i=B_1; B_2, B_3 unspecified\\ (\alpha+i\beta_2)\begin{pmatrix}1\\0\end{pmatrix}+(\beta_1+i\beta_3)\begin{pmatrix}0\\1\end{pmatrix}, i=B_2; B_1, B_3 unspecified\\ (\alpha+i\beta_3)\begin{pmatrix}1\\0\end{pmatrix}+(\beta_2+i\beta_1)\begin{pmatrix}0\\1\end{pmatrix}, i=B_3; B_1, B_2 unspecified\end{cases}$$

The above maps explicitly show how more detailed information, contained in geometric algebra states, gets lost when down-mapped to “complex” two dimensional vectors. The formulas mean that to recover which $s(\alpha, \beta, S)$ in three dimensions corresponds to qubit

$$|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix}$$

we need, firstly, to define which plane B_1 in three dimensions should be taken as “complex” plane. Secondly, choose any other plane B_2 orthogonal to B_1. After that the third plane B_3 can be selected by predefined right-hand screw space orientation:

$$I_3 B_1 I_3 B_2 I_3 B_3 = I_3 \Rightarrow B_1 I_3 B_2 I_3 B_3 = 1 \Rightarrow -B_1 I_3 I_3 B_2 B_3 = 1 \Rightarrow B_1 B_2 B_3 = 1$$

After all that we have:

$$|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \Rightarrow x_1 + y_1 B_1 + y_2 B_2 + x_2 B_3$$

with an arbitrary triple of basis bivectors satisfying multiplication rules mentioned before and the orientation condition. In that sense we can speak about fiber bundle $G_3^+ \to C^2$ as the set

$$\left(G_3^+ (total\_space), C^2 (base\_space), \pi (projection), F \right)$$

The standard fiber F is group of rotations of the triple of basis bivectors, the group elements are states from G_3^+. The projection is defined by the first option in slide 18.

Lifting of the qubit pure states to g-qubits

Conventional quantum mechanical qubit is linear combination of basis states with complex valued coefficients:

$$|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We have the map

$$|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \Rightarrow x_1 + y_1 B_1 + y_2 B_2 + x_2 B_3 = x_1 + y_1 B_1 + (x_2 + y_2 B_1) B_3$$

where the triple of basis bivectors can be arbitrary rotated as a whole.

That means that the first component of the qubit quantum mechanical state get fiber (lift) in G_3^+:

$$\begin{pmatrix} x_1 + iy_1 \\ 0 \end{pmatrix} \Rightarrow x_1 + y_1 B_1$$

with a unit bivector B_1 arbitrary oriented in three dimensions. The action of such g-qubit, state, on a bivector (for some simplicity) observable $c_1B_1 + c_2B_2 + c_3B_3$ gives the result (result of measurement):

$$(x_1 - y_1B_1)(c_1B_1 + c_2B_2 + c_3B_3)(x_1 + y_1B_1) = \left[c_1\left(x_1^2 + y_1^2\right)\right]B_1 +$$
$$\left[c_2\left(x_1^2 - y_1^2\right) - 2x_1y_1c_3\right]B_2 + \left[2x_1y_1c_2 + c_3\left(x_1^2 - y_1^2\right)\right]B_3 =$$
$$c_1B_1 + (c_2\cos 2\varphi - c_3\sin 2\varphi)B_2 + (c_2\sin 2\varphi + c_3\cos 2\varphi)B_3$$

Geometrically that means that the first basis component of the observable does not change and the remaining part is rotated by angle 2φ in hyperplane expanded by basis bivectors B_2 and B_3 .

In the similar way:

$$\begin{pmatrix} 0 \\ x_2 + iy_2 \end{pmatrix} \Rightarrow (x_2 + y_2 B_1) B_3$$

with B_1 arbitrary oriented in three dimensions and B_3 orthogonal to it. Then:

$$\begin{aligned} &-B_3(x_2 - y_2 B_1)(c_1 B_1 + c_2 B_2 + c_3 B_3)(x_2 + y_2 B_1) B_3 = \\ &-B_3(c_1 B_1 + (c_2 \cos 2\psi - c_3 \sin 2\psi) B_2 + (c_2 \sin 2\psi + c_3 \cos 2\psi) B_3) B_3 = \\ &-c_1 B_1 - (c_2 \cos 2\psi - c_3 \sin 2\psi) B_2 + (c_2 \sin 2\psi + c_3 \cos 2\psi) B_3 \end{aligned}$$

The formulas for lifts of the two basis quantum mechanical states to G_3^+ explicitly show that the only difference between the lift of $|0\rangle$ and lift of $|1\rangle$ is that measurement of an observable with the second lift state additionally flips B_1 and B_2 bivectors over B_3 .

The above two results show what the G_3^+ counterparts of quantum mechanical states do reveal. Any g-qubit can be written as the state from G_3^+, associated with a general quantum mechanical qubit. Since the g-qubit plane (the bivector part) can be taken as the only basis bivector (the "complex" plane can vary in three dimensions) any quantum mechanical qubit can be written as the state $|0\rangle$ qubit because from

$$|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \Rightarrow x_1 + y_1 B_1 + y_2 B_2 + x_2 B_3 =$$

$$x_1 + \sqrt{y_1^2 + y_2^2 + x_2^2}\left(\frac{y_1}{\sqrt{y_1^2 + y_2^2 + x_2^2}} B_1 + \frac{y_2}{\sqrt{y_1^2 + y_2^2 + x_2^2}} B_2 + \frac{x_2}{\sqrt{y_1^2 + y_2^2 + x_2^2}} B_3 \right)$$

we get:

$$|\psi\rangle = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} \Rightarrow x_1 + I_B\sqrt{y_1^2 + y_2^2 + x_2^2}$$

and this g-qubit is projected onto C^2 with the result

$$x_1 + I_B\sqrt{y_1^2 + y_2^2 + x_2^2} \Rightarrow \begin{pmatrix} x_1 + I_B\sqrt{y_1^2 + y_2^2 + x_2^2} \\ 0 \end{pmatrix}$$

where

$$I_B = \frac{y_1}{\sqrt{y_1^2 + y_2^2 + x_2^2}} B_1 + \frac{y_2}{\sqrt{y_1^2 + y_2^2 + x_2^2}} B_2 + \frac{x_2}{\sqrt{y_1^2 + y_2^2 + x_2^2}} B_3$$

If we take any two bivectors I_B^1 and I_B^2 orthogonal to I_B and to each other, so that the resulting basis bivector triple matches selected, right screw, orientation

$$I_B I_B^1 I_B^2 = 1$$

then the states

$$I_B I_B^i, i = 1,2$$

when acting on an observable I_B, flip the I_B because

$$-I_B^1 I_B I_B^1 = I_B^1 I_B^2 = -I_B$$

and

$$-I_B^2 I_B I_B^2 = -I_B^2 I_B^1 = -I_B$$

Conclusion

The difference between the two types of states revealed with geometrical algebra counterparts of common quantum mechanical qubit basic states is that one of the states, in addition to the observable rotation in the plane explicitly defined by g-qubit, flips the result over that plane.

G-qubits as points on the unit radius sphere in four dimensions

The g-qubits as unit value elements of the geometric algebra G_3^+ can be conveniently thought about as points on the unit radius sphere S^3 in four dimensions:

$$S^3 = \{(x, y, z, t) : x^2 + y^2 + z^2 + t^2 = 1\}$$

The states are elements of the form:

$$\alpha + I_S \beta = \alpha + \beta(b_1 B_1 + b_2 B_2 + b_3 B_3) \equiv \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$$

of the unit value:

$$\alpha^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1$$

so they can be considered as points on S^3.

There exists a notion of *Clifford translation* for the quantum mechanical qubit states: $S^3 \ni |\psi\rangle \to e^{i\varphi}|\psi\rangle$, φ is real scalar, the states considered as S^3 points since:

$$|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, z_1^2 + z_2^2 = z_1 \tilde{z}_1 + z_2 \tilde{z}_2 = 1, z_k = z_k^1 + i z_k^2, k = 1,2$$

It is called *translation* particularly because it does not change distances between points when applied to different points on the sphere.

In geometric algebra terms imaginary unit has explicit geometrical identification as bivector in three dimensions and Clifford translation gets generalization:

$$s(\alpha,\beta,S) \equiv \alpha + I_S \beta \to (\cos\varphi + I_{S_{Cl}} \sin\varphi) s(\alpha,\beta,S) \equiv e^{I_{S_{Cl}}\varphi} s(\alpha,\beta,S)$$

The plane of the angle φ, namely S_{Cl}, in this translation is not generally the same as the plane S of the operand $s(\alpha, \beta, S)$.

The tangent of transformation $s(\alpha, \beta, S) \rightarrow e^{I_{S_{Cl}}\varphi} s(\alpha, \beta, S)$ (I will continue call it *generalized Clifford translation, GCT)* at $\varphi = 0$ in the direction of φ in the plane S_{Cl} is:

$$\frac{\partial}{\partial \varphi} e^{I_{S_{Cl}}\varphi} s(\alpha, \beta, S)\Big|_{\varphi=0} = I_{S_{Cl}} s(\alpha, \beta, S)$$

Let's expand the g-qubit bivector in the basis with $I_{S_{Cl}}$ taken as the first basis bivector:

$$\alpha + I_S \beta = \alpha + \beta(b_1 B_1 + b_2 B_2 + b_3 B_3) = \alpha + \beta\left(b_1^{Cl} I_{S_{Cl}} + b_2^{Cl} B_2^{Cl} + b_3^{Cl} B_3^{Cl}\right) =$$

$$\alpha + \beta_1^{Cl} I_{S_{Cl}} + \beta_2^{Cl} B_2^{Cl} + \beta_3^{Cl} B_3^{Cl}$$

Unit value bivectors B_2^{Cl} and B_3^{Cl} are constructed from unit bivector $I_{S_{Cl}}$ as described in slide 19.

The tangent

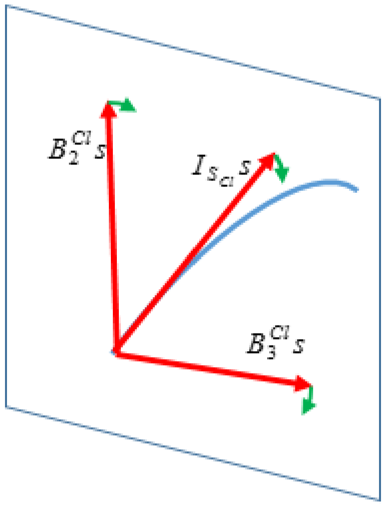
$$s(\alpha,\beta,S)I_{S_{Cl}} = I_{S_{Cl}}\left(\alpha + \beta_1^{Cl} I_{S_{Cl}} + \beta_2^{Cl} B_2^{Cl} + \beta_3^{Cl} B_3^{Cl}\right) =$$
$$-\beta_1^{Cl} + \alpha I_{S_{Cl}} + \beta_3^{Cl} B_2^{Cl} - \beta_2^{Cl} B_3^{Cl}$$

is orthogonal to

$$s(\alpha,\beta,S) = \alpha + \beta_1^{Cl} I_{S_{Cl}} + \beta_2^{Cl} B_2^{Cl} + \beta_3^{Cl} B_3^{Cl}$$

in the sense of usual scalar product as the sum of basis components products. The tangent is the velocity at $\varphi = 0$ along the orbit $e^{I_{S_{Cl}}\varphi} s(\alpha,\beta,S)$ which is big circle on the S^3 sphere lying in the plane of bivector $I_{S_{Cl}}$.

Two other tangents of the three dimensional tangent space at $\varphi = 0$ are $B_2^{Cl}s(\alpha,\beta,S)$ and $B_3^{Cl}s(\alpha,\beta,S)$. They are orthogonal to the state $s(\alpha,\beta,S)$, to the translational velocity $I_{S_{Cl}}s(\alpha,\beta,S)$ and to each other. The instant rotation velocities of the three tangents are equal in value. Tangents $B_2^{Cl}s(\alpha,\beta,S)$ and $B_3^{Cl}s(\alpha,\beta,S)$ rotate in their own hyperplane with the same, by value, instant velocities as the velocity along Clifford orbit:



GCT in G_3^+ is a kind of generalization of the states transformations by Hamiltonians in conventional quantum mechanics. A Hamiltonian is called there *observable* and is expandable in the basis of the Pauli's matrices. Pauli's matrices correspond to vectors in 3D in the geometric algebra terms (with some necessary sign and order modifications to strictly keep up with selected space orientation) and there exists explicit mapping between the geometrical objects, vectors, and formal matrices. Though a Hamiltonian as self-adjoint matrix, of second order in the considered qubit case, is not element of G_3^+, its action on a qubit, after lifting everything to G_3^+, is multiplication of corresponding g-qubit by an exponent:

$$|\psi\rangle \to H|\psi\rangle \Rightarrow s(\alpha,\beta,S) \to e^{I_{S(H)}\varphi(H)} s(\alpha,\beta,S)$$

where plane $S(H)$ and angle $\varphi(H)$ depend on the Hamiltonian matrix in a tricky, though analytically clear way.

In the conventional quantum mechanics Hamiltonian of a qubit is self adjoint matrix of general form:

$$H = \begin{pmatrix} \alpha + \beta_1 & \beta_2 - i\beta_3 \\ \beta_2 + i\beta_3 & \alpha - \beta_1 \end{pmatrix}$$

The matrix lift to G_3 (not G_3^+) is $\alpha + I_3(\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3)$ where I_3 is oriented volume in three dimensions and, otherwise arbitrary, basis bivectors satisfy multiplication rules:

$$B_1 B_2 = -B_3 \qquad B_1 B_3 = B_2 \qquad B_2 B_3 = -B_1$$

Due to the critical differences between the geometric algebra formalism, with its varying g-qubit plane, and conventional quantum mechanics formalism not specifying complex plane, the Hamiltonian lift in G_3 can be written as:

$\alpha + I_3 B_H$, where $B_H = \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$

Below I will only consider the case of Hamiltonian with zero trace:

$$H = \begin{pmatrix} \beta_1 & \beta_2 - i\beta_3 \\ \beta_2 + i\beta_3 & -\beta_1 \end{pmatrix}$$

Its lift to G_3 is

$$gH \equiv I_3(\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3)$$

The geometric algebra plane in three dimensions associated with conventional Hamiltonian generally varies with time.

Let’s make infinitesimal GCT of some G_3^+ state in the instant plane of B_H :

$$s(\alpha,\beta,S) \rightarrow e^{-I_3 \frac{gH(t)}{|gH(t)|}\left(|gH(t)|dt\right)} s(\alpha,\beta,S)$$

The exponent factor $-I_3 \frac{gH(t)}{|gH(t)|}$ is unit value bivector of the geometric algebra Hamiltonian plane, instant “complex” plane, and $|gH(t)|dt$ is angle of infinitesimal rotation in that plane.

If the Hamiltonian plane B_H does not explicitly depend on time (the assumption is close to unspecified complex plane in conventional quantum mechanics) we get from the above transformation with a finite time value:

$$\frac{\partial}{\partial t} e^{-I_3 \frac{gH}{|gH|}(|gH| t)} s(\alpha, \beta, S) = -I_3 gH e^{-I_3 \frac{gH}{|gH|}(|gH| t)} s(\alpha, \beta, S)$$

or

$$I_3 \frac{\partial}{\partial t} e^{-I_3 \frac{gH}{|gH|}(|gH| t)} s(\alpha, \beta, S) = gH e^{-I_3 \frac{gH}{|gH|}(|gH| t)} s(\alpha, \beta, S)$$

which is exactly the Schrodinger equation for a state evolving along the GCT orbit.

Now suppose that the plane of Hamiltonian is varying. Then

$$\frac{\partial}{\partial t} e^{-I_3 \frac{gH(t)}{|gH(t)|}(|gH(t)|t)} s(\alpha,\beta,S) = -I_3 gH(t) \frac{\partial}{\partial t} gH(t) e^{-I_3 \frac{gH(t)}{|gH(t)|}(|gH(t)|t)} s(\alpha,\beta,S) =$$

$$-I_3 gH(t) I_3 \left(\sum_{i=1}^{3} \frac{\partial \beta_i}{\partial t} B_i \right) e^{-I_3 \frac{gH(t)}{|gH(t)|}(|gH(t)|t)} s(\alpha,\beta,S)$$